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12

31 X

200 - 11 - 22

198 - 221 6 - 13
199 1 - 10

202 - 9 - 17
7 - 14, 15, 20

223 11 - 15
237 5 - 12

204 top of page

1,72

14-2

206

18-

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222-3-12

23-32

6

11-18

Page 206

207

208

Deborah Johnson
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Page 210 45-53 — By m.c.

11 16-26

239 — 3, 5, 7, 9, 11, 13, 15

16.

158 - 28 - 31

153 25, 27, 29

ESSENTIALS OF ALGEBRA

FOR

SECONDARY SCHOOLS

BY

WEBSTER WELLS, S.B.

PROFESSOR OF MATHEMATICS IN THE MASSACHUSETTS
INSTITUTE OF TECHNOLOGY

* * *

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PREFACE.

THE cordial reception which the author's other Algebras have received at the hands of the educational public, their extensive use in schools of the highest rank in all parts of the country, the appreciative recommendations which have come to him from instructors of reputation, lead him to believe that this latest attempt to adequately meet the demands of the best secondary schools will be cordially welcomed.

Our teachers are progressive, and the author who fails to keep abreast of the times, and in sympathy with the best educational thought and methods, will appeal in vain for the patronage and sympathies of his fellow-teachers.

Fully conscious of the above truth, the author earnestly recommends "The Essentials of Algebra" to the attention of the educational public.

It affords a thorough and complete treatment of elementary Algebra, and attention is especially invited to the following features: —

The introduction of easy problems at the very outset; § 5.

The Addition and Multiplication of Positive and Negative Numbers; §§ 14 to 19.

The Addition of Similar Terms; § 31.

The discussion of Simple Equations, not involving Fractions, directly after Division; Chap. VII.

The suggestions in regard to the solution of problems; §§ 76, 77.

The discussion of the theoretical principles involved in the handling of fractions; §§ 129, 136, 143, 145.

The examples on page 176.

The discussion of square roots and cube roots of arithmetical numbers; §§ 197, 198, 203, 204.

The examples at the end of § 229.

The solution of equations by factoring; §§ 266, 267.

The factoring of a quadratic expression when the coefficient of x^2 is a perfect square; § 286.

Great care has been taken to state the various definitions and rules with accuracy, and every principle has been demonstrated with strict regard to the logical principles involved. As a rule, no definition has been introduced until its use became necessary.

The examples and problems have been selected with great care, are ample in number, and thoroughly graded. They are especially numerous in the important chapters on Factoring, Fractions, and Radicals.

The latest English practice has been followed in writing Arithmetic, Geometric, and Harmonic, for Arithmetical, Geometrical, and Harmonical, in the progressions.

The author wishes to acknowledge, with hearty thanks, the many suggestions and the assistance that he has received from principals and teachers of secondary schools in all parts of the country, in improving and perfecting the work.

WEBSTER WELLS.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
March, 1897.

CONTENTS.

PAGE

I. DEFINITIONS AND NOTATION 1
Solution of Problems by Algebraic Methods 2
Algebraic Expressions 6

II. POSITIVE AND NEGATIVE NUMBERS 9
Addition of Positive and Negative Numbers 11
Multiplication of Positive and Negative Numbers . . 12

III. ADDITION AND SUBTRACTION OF ALGEBRAIC EXPRESSIONS . 15
Addition of Monomials 16
Addition of Polynomials 20
Subtraction 21
Subtraction of Monomials 22
Subtraction of Polynomials 23

IV. PARENTHESES 26
Removal of Parentheses 26
Introduction of Parentheses 28

V. MULTIPLICATION 29
Multiplication of Monomials 30
Multiplication of Polynomials by Monomials 32
Multiplication of Polynomials by Polynomials . . . 32

VI. DIVISION 37
Division of Monomials 38
Division of Polynomials by Monomials 39
Division of Polynomials by Polynomials 40

VII. SIMPLE EQUATIONS 48
Properties of Equations 49
Solution of Simple Equations 50
Problems 52

VIII. IMPORTANT RULES IN MULTIPLICATION AND DIVISION . . 59

IX. FACTORING 67

PAGE

X. HIGHEST COMMON FACTOR 81

XI. LOWEST COMMON MULTIPLE 91

XII. FRACTIONS 96
- Reduction of Fractions 97
- Addition and Subtraction of Fractions 105
- Multiplication of Fractions 111
- Division of Fractions 113
- Complex Fractions 115

XIII. SIMPLE EQUATIONS (*Continued*) 120
- Solution of Equations containing Fractions . . . 120
- Solution of Literal Equations 124
- Solution of Equations involving Decimals 126
- Problems 127
- Problems involving Literal Equations 135

XIV. SIMULTANEOUS EQUATIONS
- Containing Two Unknown Quantities 138

XV. SIMULTANEOUS EQUATIONS
- Containing more than Two Unknown Quantities. . 150

XVI. PROBLEMS
- Involving Simultaneous Equations 154

XVII. INEQUALITIES 165

XVIII. INVOLUTION 170
- Involution of Monomials 170
- Square of a Polynomial 171
- Cube of a Binomial 172

XIX. EVOLUTION 174
- Evolution of Monomials 174
- Square Root of a Polynomial 176
- Square Root of an Arithmetical Number 179
- Cube Root of a Polynomial 183
- Cube Root of an Arithmetical Number 186

XX. THEORY OF EXPONENTS 191

XXI. RADICALS 201
- Reduction of a Radical to its Simplest Form . . . 201
- Addition and Subtraction of Radicals 205
- To Reduce Radicals of Different Degrees to Equivalent Radicals of the Same Degree 206

XXI. RADICALS (*Continued*). PAGE
Multiplication of Radicals 207
Division of Radicals 210
Involution of Radicals 212
Evolution of Radicals 212
To Reduce a Fraction having an Irrational Denominator to an Equivalent Fraction whose Denominator is Rational 213
Properties of Quadratic Surds 215
Imaginary Numbers 218
Solution of Equations containing Radicals . . . 222

XXII. QUADRATIC EQUATIONS 224
Pure Quadratic Equations 224
Affected Quadratic Equations 226
Problems 238

XXIII. EQUATIONS SOLVED LIKE QUADRATICS 243
Equations in the Quadratic Form 243

XXIV. SIMULTANEOUS EQUATIONS
Involving Quadratics 248
Problems 258

XXV. THEORY OF QUADRATIC EQUATIONS 261
Factoring 263
Discussion of the General Equation 268

XXVI. ZERO AND INFINITY 270
Variables and Limits 270
The Problem of the Couriers 272

XXVII. INDETERMINATE EQUATIONS 275

XXVIII. RATIO AND PROPORTION 278
Properties of Proportions 279

XXIX. VARIATION 287

XXX. PROGRESSIONS 291
Arithmetic Progression 291
Geometric Progression 299
Harmonic Progression 307

XXXI. THE BINOMIAL THEOREM 310
Positive Integral Exponent 310

PAGE
XXXII. UNDETERMINED COEFFICIENTS 317
Convergency and Divergency of Series 317
The Theorem of Undetermined Coefficients . . . 320
Expansion of Fractions into Series 321
Expansion of Radicals into Series 323
Partial Fractions 324
Reversion of Series 330

XXXIII. THE BINOMIAL THEOREM 332
Fractional and Negative Exponents 332

XXXIV. LOGARITHMS 339
Properties of Logarithms 341
Use of the Table 346
Applications 351
Arithmetical Complement 353
Exponential Equations 357

ANSWERS TO THE EXAMPLES.

ALGEBRA.

I. DEFINITIONS AND NOTATION.

1. In **Algebra,** the operations of Arithmetic are abridged and generalized by means of **Symbols.**

2. Symbols which represent Numbers.

The symbols generally employed to represent numbers are the *figures of Arithmetic* and the *letters of the Alphabet.*

Known Numbers are usually represented by the first letters of the alphabet, as a, b, c.

Unknown Numbers, or those whose values are to be determined, are usually represented by the last letters of the alphabet, as x, y, z.

3. Symbols which represent Operations.

The following symbols have the same meaning in Algebra as in Arithmetic:

$+$, read "*plus.*"
$-$, read "*minus.*"
$\times$, read "*times,*" "*into,*" or "*multiplied by.*"
$\div$, read "*divided by.*"

The sign of multiplication is usually omitted in Algebra, except between arithmetical figures.

Thus, $2 \times x$ is written $2x$.

Division is usually indicated by a horizontal line.

Thus, $a \div b$ is written $\frac{a}{b}$.

4. The **Sign of Equality**, =, is read *"equals,"* or *"is equal to."*

An **Equation** is a statement that two numbers are equal.

SOLUTION OF PROBLEMS BY ALGEBRAIC METHODS.

5. The following examples will illustrate the use of Algebraic symbols in the solution of problems.

The utility of the process consists in the fact that the unknown numbers are represented by *symbols*, and that the various operations are stated in *Algebraic language.*

1. The sum of two numbers is 30, and the greater exceeds the less by 4; what are the numbers?

We will represent the less number by x.

Then the greater will be represented by $x + 4$.

By the conditions of the problem, the sum of the greater number and the less is 30; this is stated in Algebraic language as follows:

$$x + 4 + x = 30. \qquad (1)$$

But the sum of x and x is twice x, or $2x$; whence, equation (1) may be written

$$2x + 4 = 30.$$

Now if $2x$ plus 4 equals 30, $2x$ must equal $30 - 4$, or 26.

Whence, $2x = 26.$

But if twice x is 26, x must be one-half of 26, or 13.

Hence, the less number is 13, and the greater is $13 + 4$, or 17.

The written work will stand as follows:

Let $x =$ the less number.

Then, $x + 4 =$ the greater number.

By the conditions, $x + 4 + x = 30.$

Or, $2x + 4 = 30.$

Whence, $2x = 26.$

Dividing by 2, $x = 13$, the less number.

Whence, $x + 4 = 17$, the greater number.

2. The sum of the ages of A and B is 109 years, and A is 13 years younger than B; find their ages.

Let x represent the number of years in B's age.

Then, $x - 13$ will represent the number of years in A's age.

By the conditions of the problem, the sum of the ages of A and B is 109 years.

Whence, $x + x - 13 = 109.$

Or, $2x - 13 = 109.$

Now if $2x$ minus 13 equals 109, $2x$ must equal $109 + 13$, or 122.

Whence, $2x = 122.$

Dividing by 2, $x = 61$, the number of years in B's age.

And, $x - 13 = 48$, the number of years in A's age.

The written work will stand as follows:

Let $x =$ the number of years in B's age.

Then, $x - 13 =$ the number of years in A's age.

By the conditions, $x + x - 13 = 109.$

Or, $2x - 13 = 109.$

Whence, $2x = 122.$

Dividing by 2, $x = 61$, the number of years in B's age.

Therefore, $x - 13 = 48$, the number of years in A's age.

3. A, B, and C together have $66. A has one-half as much as B, and C has 3 times as much as A. How much has each?

Let $x =$ the number of dollars A has.

Then, $2x =$ the number of dollars B has,

and $3x =$ the number of dollars C has.

By the conditions,

$$x + 2x + 3x = 66.$$

But the sum of x, twice x, and 3 times x is 6 times x, or $6x$.

Whence, $6x = 66.$

Dividing by 6, $x = 11$, the number of dollars A has.

Whence, $2x = 22$, the number of dollars B has,

and $3x = 33$, the number of dollars C has.

PROBLEMS.

4. The greater of two numbers is 4 times the less, and their sum is 70. What are the numbers?

5. The sum of the ages of A and B is 116 years, and A is 18 years younger than B. What are their ages?

6. Divide 123 into two parts, such that the greater exceeds the less by 67.

7. The sum of the ages of A and B is 102 years, and A is 26 years older than B. What are their ages?

8. Divide $ 93 between A and B, so that A may receive $ 23 less than B.

9. Divide $ 56 between A and B, so that A may receive 6 times as much as B.

10. Divide 85 into two parts, one of which shall be 19 less than the other.

11. Divide $ 72 between A and B, so that A may receive one-third as much as B.

12. A certain hall contains 425 persons; there are 3 times as many men as women, and 4 times as many women as children. How many are there of each?

13. A man had $ 4.95. After spending a certain sum, he found that he had left 4 times as much as he had spent. How much did he spend?

14. A, B, and C together have $ 96. B has twice as much money as C, and A has as much as B and C together. How much has each?

15. The sum of three numbers is 168. The second is 23 less than the first, and the third is 3 times the second. What are the numbers?

16. A, B, and C together have $ 230. A has $ 21 more than B, and $ 17 less than C. How much has each?

17. A watch and chain are together worth $ 56, and the chain is worth one-sixth as much as the watch. What is the value of each?

18. Divide 169 into three parts, the first of which is one-half of the second, and the second one-fifth of the third.

19. Divide $ 144 into three parts such that the second is one-third of the first, and one-fourth of the third.

20. A man bought a cow, a sheep, and a hog for $ 84. The price of the hog was one-fifth the price of the cow, and $ 7 less than the price of the sheep. What was the price of each?

21. The sum of three numbers is 127. The first is one-half of the third, and 17 greater than the second. What are the numbers?

22. At a certain election, two candidates, A and B, together received 508 votes; and A had a majority of 136. How many did each receive?

23. The sum of the ages of A, B, and C is 101 years. A is 17 years younger than B, and 15 years older than C. What are their ages?

24. Divide $ 174 among A, B, and C, so that A may receive 4 times as much as B, and $ 42 more than C.

25. My horse, carriage, and harness are together worth $ 456. The carriage is worth 8 times as much as the harness, and $ 48 less than the horse. Find the value of each.

26. Divide $ 155 into three parts such that the first shall be 5 times the second, and one-fifth of the third.

27. At a certain election, three candidates, A, B, and C, together received 512 votes. A received 28 less than B, and 64 less than C. How many did each receive?

28. Divide $ 69 among A, B, C, and D, so that A may receive $ 5 more than B, C as much as A and B together, and D as much as A and C together.

29. The sum of four numbers is 160. The first is 3 times the second, the second 3 times the third, and the third 3 times the fourth. What are the numbers?

DEFINITIONS.

6. If a number be multiplied by itself any number of times, the result is called a *power* of that number.

An **Exponent** is a number written at the right of, and above another number, to indicate what power of the latter is to be taken.

Thus,

a^2, read "*a square*," or "*a second power*," denotes $a \times a$;
a^3, read "*a cube*," or "*a third power*," denotes $a \times a \times a$;
a^4, read "*a fourth*," or "*a fourth power*," denotes $a \times a \times a \times a$, and so on.

If no exponent is expressed, the *first* power is understood.

Thus, a is the same as a^1.

7. Symbols of Aggregation.

The *parentheses* ( ), the *brackets* [ ], the *braces* { }, and the *vinculum* ——, indicate that the numbers enclosed by them are to be taken collectively; thus,

$$(a+b) \times c,\ [a+b] \times c,\ \{a+b\} \times c, \text{ and } \overline{a+b} \times c$$

all indicate that the result obtained by adding b to a is to be multiplied by c.

ALGEBRAIC EXPRESSIONS.

8. An **Algebraic Expression**, or simply an **Expression**, is a number expressed in algebraic symbols; as,

$$2,\ a, \text{ or } 2x^2 - 3ab + 5.$$

The **Numerical Value** of an expression is the result obtained by substituting particular numerical values for the letters involved in it, and performing the operations indicated.

1. Find the numerical value of the expression

$$4a+\frac{6c}{b}-d^3,$$

when $a=4$, $b=3$, $c=5$, and $d=2$.

We have, $$4a+\frac{6c}{b}-d^3=4\times 4+\frac{6\times 5}{3}-2^3$$
$$=16+10-8=18, \text{ Ans.}$$

EXAMPLES.

Find the numerical value of each of the following when $a=3$, $b=5$, $c=2$, $d=4$, $m=4$, and $n=3$:

2. ad^2-bc^3.
3. $3abcd$.
4. $4a^2d-5bc-6cd$.
5. $\frac{ab}{c}+\frac{cd}{a}$.
6. $\frac{1}{a}-\frac{1}{b}+\frac{1}{c}$.
7. $\frac{5a^m}{3b^n}$.
8. $a^m d^n$.
9. $\frac{a}{b}+\frac{b}{c}-\frac{c}{a}$.
10. $8a^c-9c^a$.
11. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$.
12. $\frac{a^3b}{2c^2}-\frac{ab^3}{4d^2}$.
13. $\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{d^2}$.

If the expression involves *parentheses*, the operations indicated *within* the parentheses must be performed first.

14. Find the numerical value, when $a=9$, $b=7$, and $c=4$, of

$$(a-b)(b+c)-\frac{a+b}{b-c}.$$

We have, $a-b=2$, $b+c=11$, $a+b=16$, and $b-c=3$.

Then the numerical value of the expression is

$$2\times 11-\frac{16}{3}=22-\frac{16}{3}=\frac{50}{3}. \text{ Ans.}$$

Find the numerical value of each of the following expressions, when $a=5$, $b=3$, $c=4$, and $d=2$:

15. $\left(\frac{a+b}{a-b}\right)^2.$

16. $(a^2+b^2-c^2)^2.$

17. $3a^2(c-d)-2b^2(c+d).$

18. $5(a+b)^2-9(c-d)^2.$

19. $(a-b)(b+c)-(c-d)(d+a).$

20. $(a-b+c-d)(a+b-c-d).$

21. $\frac{3a-5b+7c}{4a-3b+6c}.$

22. $\frac{a+b}{b+c}+\frac{b+c}{c+d}+\frac{c+d}{d+a}.$

Find the numerical value of each of the following expressions, when $a=\frac{5}{4}$, $b=\frac{1}{2}$, $c=\frac{2}{3}$, and $x=3$:

23. $\frac{a+b}{a-b}-\frac{a-b}{a+b}.$

24. $\frac{16a-18b+15c}{44a-32b-27c}.$

25. $x^3-(4a-5c)x^2+(2a+b)x-12abc.$

26. $x^3+\left(\frac{1}{b}-\frac{1}{a}\right)x^2-\left(\frac{5}{a}+\frac{3}{b}-\frac{2}{c}\right)x-\frac{4}{c^2}.$

9. Axioms.

An **Axiom** is a truth which is assumed as self-evident.

Algebraic operations are based upon the following axioms:

1. *If the same operation be performed upon equal numbers, the resulting numbers will be equal.*

2. *If the same number be both added to, and subtracted from another, the value of the latter will not be changed.*

3. *If a number be both multiplied and divided by another, the value of the former will not be changed.*

4. *Numbers which are equal to the same number, are equal to each other.*

II. POSITIVE AND NEGATIVE NUMBERS.

10. Many concrete magnitudes are capable of existing in two opposite states.

Thus, in financial transactions, we may have *gains*, or *losses;* in the thermometer, we may have temperatures *above* zero, or *below* zero; a place on the surface of the earth may be in *north* latitude, or *south* latitude; etc.

The signs + and −, besides indicating the operations of addition and subtraction, are also used in Algebra to distinguish between the opposite states of magnitudes like the above.

Thus, in financial transactions, we may indicate *gains* or *assets* by the sign +, and *losses* or *debts* by the sign −; for example, the statement that a man's property is − $100, means that he has debts or liabilities to the amount of $100.

Again, in the thermometer, we may indicate temperatures *above* zero by the sign +, and temperatures *below* zero by the sign −; for example, + 25° means 25° above zero, and − 10° means 10° below zero.

Also, we may indicate *north* latitude and *west* longitude by the sign +, and *south* latitude and *east* longitude by the sign −; thus, a place in latitude − 30°, longitude + 95°, would be in latitude 30° south of the equator, and in longitude 95° west of Greenwich.

EXERCISES.

11. 1. At 7 A.M. the temperature is − 13°; at noon it is 8° warmer, and at 6 P.M. it is 5° colder than at noon. Required the temperatures at noon and at 6 P.M.

2. At 7 A.M. the temperature is + 6°; at noon it is 14° colder, and at 6 P.M. it is 8° warmer than at noon. Required the temperatures at noon and at 6 P.M.

3. What is the difference in latitude between two places whose latitudes are + 67° and − 48°?

4. A man has bills receivable to the amount of $ 480, and bills payable to the amount of $ 925; how much is he worth?

5. A vessel sails from the equator due north 28°, and then due south 57°; what is her latitude at the end of the voyage?

6. At 7 A.M. the temperature is − 7°, and at noon + 9°. How many degrees warmer is it at noon than at 7 A.M.?

7. What is the difference in longitude between two places whose longitudes are + 29° and − 86°?

8. The temperature at 6 A.M. is + 14°; and during the morning it grows colder at the rate of 4° an hour. Required the temperatures at 9 A.M., at 10 A.M., and at noon.

12. Positive and Negative Numbers.

If the positive and negative states of any concrete magnitude be expressed *without reference to the unit,* the results are called *positive* and *negative numbers,* respectively.

Thus, in + $ 5 and − $ 3, + 5 is a positive number, and − 3 is a negative number.

For this reason the sign + is called the *positive* sign, and the sign − the *negative* sign.

If no sign is expressed, the number is understood to be positive; thus, 5 is the same as + 5.

The negative sign can never be omitted before a negative number.

13. The **Absolute Value** of a number is the number taken independently of the sign affecting it.

Thus, the absolute value of − 3 is 3.

ADDITION OF POSITIVE AND NEGATIVE NUMBERS.

14. The result of Addition is called the *Sum*.

We shall retain for Addition in Algebra its arithmetical meaning, *so long as the numbers to be added are positive.*

We may then attach any meaning we please to addition involving other forms of number, provided the new meaning is not inconsistent with principles which have been previously established.

15. If a man gains \$5, and then loses \$3, he will be worth \$2.

If he owes \$5, and then gains \$3, he will be in debt to the amount of \$2.

If he owes \$5, and then incurs a debt of \$3, he will be in debt to the amount of \$8.

Now with the notation of § 10, losing \$3, or incurring a debt of \$3, may be regarded as adding $-\$3$ to his property.

Whence, the sum of $+\$5$ and $-\$3$ is $+\$2$;

the sum of $-\$5$ and $+\$3$ is $-\$2$;

and the sum of $-\$5$ and $-\$3$ is $-\$8$.

Or, omitting reference to the *unit*,

$$(+5)+(-3)=+2;$$
$$(-5)+(+3)=-2;$$
$$(-5)+(-3)=-8.$$

We then have the following rules:

To add a positive and a negative number, subtract the less absolute value (§ 13) *from the greater, and prefix to the result the sign of the number having the greater absolute value.*

To add two negative numbers, add their absolute values, and prefix a negative sign to the result.

16. **1.** Find the sum of $+10$ and -3.

Subtracting 3 from 10, the result is 7.

Whence, $(+10)+(-3)=+7$, *Ans.*

2. Find the sum of -12 and $+6$.

Subtracting 6 from 12, the result is 6.

Whence, $(-12)+(+6)=-6$, *Ans.*

3. Add -9 and -5.

The sum of 9 and 5 is 14.

Whence, $(-9)+(-5)=-14$, *Ans.*

EXAMPLES.

Find the values of the following:

4. $(-7)+(-5)$.

5. $(+9)+(-4)$.

6. $(-8)+(+2)$.

7. $(+6)+(-15)$.

8. $(-11)+(-16)$.

9. $(+52)+(-37)$.

10. $(-61)+(+28)$.

11. $\left(-\frac{3}{4}\right)+\left(+\frac{5}{6}\right)$.

12. $\left(-\frac{8}{5}\right)+\left(-\frac{9}{7}\right)$.

13. $(+14\frac{1}{2})+(-10\frac{7}{8})$.

14. $(-18\frac{7}{12})+(+12\frac{4}{9})$.

15. $(+20\frac{7}{10})+(-13\frac{8}{15})$.

MULTIPLICATION OF POSITIVE AND NEGATIVE NUMBERS.

17. The terms *Multiplicand, Multiplier,* and *Product* have the same meaning in Algebra as in Arithmetic.

We shall retain for Multiplication, in Algebra, its arithmetical meaning, *so long as the multiplier is a positive number.*

That is, to multiply a number by a positive integer is to *add* the first number as many times as there are units in the second.

For example, to multiply -4 by 3, we add -4 three times.

That is, $(-4) \times (+3) = (-4) + (-4) + (-4) = -12$.

We may then attach any meaning we please to multiplication by a *negative number*.

18. In Arithmetic, the product of two numbers is the same in whatever *order* they are taken.

Thus, 3×5 and 5×3 are each equal to 15.

If we assume this law to hold universally, we have

$$(+3) \times (-4) = (-4) \times (+3).$$

But by § 17, $(-4) \times (+3) = -12 = -(3 \times 4)$.

Whence, $(+3) \times (-4) = -(3 \times 4)$. (§ 9, 4)

We then have the following definition:

To multiply a number by a negative number is to multiply it by the absolute value (§ 13) *of the multiplier, and change the sign of the result.*

Thus, to multiply $+4$ by -3, we multiply $+4$ by $+3$, giving $+12$, and change the sign of the result.

That is, $(+4) \times (-3) = -12$.

Again, to multiply -4 by -3, we multiply -4 by $+3$, giving -12 (§ 17), and change the sign of the result.

That is, $(-4) \times (-3) = +12$.

19. From §§ 17 and 18 we derive the following rule:

To multiply one number by another, multiply together their absolute values.

Make the product **plus** *when the multiplicand and multiplier are of* **like** *sign, and* **minus** *when they are of* **unlike** *sign.*

1. Multiply $+8$ by -5.

By the rule, $(+8)\times(-5)=-(8\times 5)=-40$, *Ans.*

2. Multiply -7 by -9.

By the rule, $(-7)\times(-9)=+(7\times 9)=+63$, *Ans.*

EXAMPLES.

Find the values of the following:

3. $(+6)\times(-3)$.

4. $(-10)\times(+5)$.

5. $(-7)\times(-6)$.

6. $(-12)\times(+4)$.

7. $(-8)\times(-8)$.

8. $(-15)\times(+9)$.

9. $(+11)\times(-16)$.

10. $(-24)\times(-18)$.

11. $\left(+\frac{4}{7}\right)\times\left(-\frac{5}{9}\right)$.

12. $\left(-\frac{8}{15}\right)\times\left(-\frac{9}{14}\right)$.

13. $\left(-\frac{18}{55}\right)\times\left(+\frac{22}{27}\right)$.

14. $(+9\tfrac{1}{7})\times(-2\tfrac{5}{8})$.

15. $(-1\tfrac{8}{15})\times(-1\tfrac{4}{21})$.

III. ADDITION AND SUBTRACTION OF ALGEBRAIC EXPRESSIONS.

DEFINITIONS.

20. A **Monomial**, or **Term**, is an expression (§ 8) whose parts are not separated by the signs + or −; as $2x^2$, $-3ab$, or 5.

$2x^2$, $-3ab$, and $+5$ are called the *terms* of the expression $2x^2 - 3ab + 5$.

A **Positive Term** is one preceded by a + sign; as $+5a$.

If no sign is expressed, the term is understood to be positive.

A **Negative Term** is one preceded by a − sign; as $-3ab$.

The − sign can never be omitted before a negative term.

21. If two or more numbers are multiplied together, each of them, or the product of any number of them, is called a **Factor** of the product.

Thus, a, b, c, ab, ac, and bc are factors of the product abc.

22. Any factor of a product is called the **Coefficient** of the product of the remaining factors.

Thus, in $2ab$, 2 is the coefficient of ab, $2a$ of b, a of $2b$, etc.

23. If one factor of a product is expressed in *numerals*, and the other in *letters*, the former is called the *numerical coefficient* of the latter.

Thus, in $2ab$, 2 is the numerical coefficient of ab.

If no numerical coefficient is expressed, the coefficient 1 is understood; thus, a is the same as $1a$.

24. By § 19, $(-3) \times a = -(3 \times a) = -3a$.

That is, $-3a$ is the product of -3 and a.

Then, -3 is the *numerical coefficient* of a in $-3a$.

Thus, in a negative term, the numerical coefficient includes the *sign*.

25. **Similar** or **Like Terms** are those which do not differ at all, or else differ only in their numerical coefficients; as $2x^2y$ and $-7x^2y$.

Dissimilar or **Unlike Terms** are those which are not similar; as $3x^2y$ and $3xy^2$.

ADDITION OF MONOMIALS.

26. The sum of a and b is $a + b$ (§ 3); and the sum of a and $-b$ is expressed $a + (-b)$.

27. Required the sum of a and $-b$.

By § 10, if a man incurs a debt of \$4, we may regard the transaction either as adding $-$\$4 to his property, or as subtracting \$4 from it.

That is, *adding a negative number is equivalent to subtracting a positive number of the same absolute value* (§ 13).

Thus, the sum of a and $-b$ is obtained by subtracting b from a.

Or, $$a + (-b) = a - b.$$

28. It follows from §§ 26 and 27 that *the addition of monomials is effected by uniting them with their respective signs.*

Thus, the sum of a, $-b$, c, $-d$, and $-e$ is

$$a - b + c - d - e.$$

It is immaterial in what *order* the terms are united, provided each has its proper sign.

Hence, the above result may also be expressed

$$c + a - e - d - b,$$

$$-d - b + c - e + a, \text{ etc.}$$

29. If the same number be both added to, and subtracted from another, the value of the latter will not be changed (§ 9).

That is, $a + b - b = a$.

Hence, terms of equal absolute value, but opposite sign, in an expression, neutralize each other, or *cancel.*

30. To multiply 4 by $5 + 3$, we multiply 4 by 5, and then 4 by 3, and add the second result to the first.

In like manner, to multiply a by $b + c$, we multiply a by b, and then a by c, and add the second result to the first.

That is, $a(b + c) = ab + ac$.

31. Addition of Similar Terms (§ 25).

1. Find the sum of $5a$ and $3a$.

We have,
$$\begin{aligned} 5a + 3a &= (5 + 3)a & (\S\ 30) \\ &= 8a, \textit{ Ans.} \end{aligned}$$

2. Find the sum of $-5a$ and $-3a$.

We have,
$$\begin{aligned} (-5a) + (-3a) &= (-5) \times a + (-3) \times a & (\S\ 19) \\ &= [(-5) + (-3)] \times a & (\S\ 30) \\ &= (-8) \times a & (\S\ 15) \\ &= -8a, \textit{ Ans.} & (\S\ 19) \end{aligned}$$

3. Find the sum of $5a$ and $-3a$.

We have,
$$\begin{aligned} 5a + (-3)a &= [5 + (-3)] \times a & (\S\ 30) \\ &= 2a, \textit{ Ans.} & (\S\ 15) \end{aligned}$$

4. Find the sum of $-5a$ and $3a$.

We have,
$$(-5)a + 3a = [(-5) + 3] \times a \qquad (\S\ 30)$$
$$= (-2) \times a \qquad (\S\ 15)$$
$$= -2a, \textit{ Ans.}$$

Therefore, *to add two similar terms, find the sum of their numerical coefficients* (§§ 15, 24), *and affix to the result the common letters.*

EXAMPLES.

Add the following:

5. $5a$ and $-12a$.

6. $-7m$ and $-8m$.

7. $15x$ and $-11x$.

8. $-10a^2$ and $4a^2$.

9. $-bc$ and $6bc$.

10. xyz and $-9xyz$.

11. $-18m^2n^3$ and $-27m^2n^3$.

12. $36a^3bc^2$ and $-19a^3bc^2$.

13. Required the sum of $2a$, $-a$, $3a$, $-12a$, and $6a$.

Since the order of the terms is immaterial (§ 28), we may add the positive terms first, and then the negative terms, and finally combine these two results.

The sum of $2a$, $3a$, and $6a$ is $11a$.

The sum of $-a$ and $-12a$ is $-13a$.

Then the required sum is $11a + (-13a)$, or $-2a$, *Ans.*

Add the following:

14. $9a$, $-7a$, and $8a$.

15. $13x$, $-x$, $-10x$, and $5x$.

16. $12abc$, abc, $-6abc$, and $-17abc$.

17. $15m^2$, $-11m^2$, $-4m^2$, m^2, and $14m^2$.

18. $21x^3y^4$, $-16x^3y^4$, $-x^3y^4$, $3x^3y^4$, and $-19x^3y^4$.

If the terms are not all similar, we may combine the similar terms, and unite the others with their respective signs (§ 28).

19. Required the sum of $12a$, $-5x$, $-3y^2$, $-5a$, $8x$, and $-3x$.

The sum of $12a$ and $-5a$ is $7a$.
The sum of $-5x$, $8x$, and $-3x$ is 0 (§ 29).
Then the required sum is $7a - 3y^2$, *Ans.*

Add the following:

20. $8ab$, $-7cd$, $-5ab$, and $3cd$.

21. $6x$, $-10z$, $2y$, $4z$, $-9y$, and $-x$.

22. $12m^2$, $-2m$, $-8n$, 5, $-3n$, $-7m^2$, and $11n$.

23. $10a$, $-6d$, $-5c$, $12b$, $-a$, c, $-3c$, and $-9a$.

24. $7x$, $-4y$, $-3z$, $9y$, $-2x$, $-8x$, $-5z$, $6y$, and $-z$.

DEFINITIONS.

32. A **Polynomial** is an algebraic expression consisting of more than one term; as $a + b$, or $2x^2 - 3xy - 5y^2$.

A **Binomial** is a polynomial of two terms; as $a + b$.

A **Trinomial** is a polynomial of three terms.

33. A polynomial is said to be *arranged* according to the *descending* powers of any letter, when the term containing the highest power of that letter is placed first, that having the next lower immediately after, and so on. Thus,

$$x^4 + 3x^3y - 2x^2y^2 + 3xy^3 - 4y^4$$

is arranged according to the descending powers of x.

Note. The term $-4y^4$, which does not involve x at all, is regarded as containing the lowest power of x in the above expression.

A polynomial is said to be arranged according to the *ascending* powers of any letter, when the term containing the lowest power of that letter is placed first, that having the next higher immediately after, and so on. Thus,

$$x^4 + 3x^3y - 2x^2y^2 + 3xy^3 - 4y^4$$

is arranged according to the ascending powers of y.

ADDITION OF POLYNOMIALS.

34. A polynomial may be regarded as the sum of its separate monomial terms (§ 28).

Thus, $2a - 3b + 4c$ is the sum of $2a$, $-3b$, and $4c$.

Hence, *the addition of polynomials may be effected by uniting their terms with their respective signs.*

1. Required the sum of $6a - 7x^2$, $3x^2 - 2a + 3y^3$, and $2x^2 - a - mn$.

It is convenient in practice to set the expressions down one underneath the other, similar terms being in the same vertical column.

We then add the terms in each column, and unite the results with their respective signs. Thus,

$$\begin{array}{rrrr} 6a & -7x^2 & & \\ -2a & +3x^2 & +3y^3 & \\ -\ a & +2x^2 & & -mn \\ \hline 8a & -2x^2 & +3y^3 & -mn, \end{array} \text{ Ans.}$$

EXAMPLES.

Add the following:

2.

$$\begin{array}{rr} 7a & -5b \\ -9a & +2b \\ 3a & -\ b \\ \hline \end{array}$$

3.

$$\begin{array}{rr} -\ 8m^2 & +\ 5n^3 \\ 12m^2 & -16n^3 \\ -\ 6m^2 & +14n^3 \\ \hline \end{array}$$

4.

$$\begin{array}{rr} -19ab & -\ 7cd \\ 8ab & -17cd \\ 5ab & +13cd \\ \hline \end{array}$$

5. $4a - 6b + 3c$ and $5a + 2b - 9c$.

6. $m^2 + 2mn + n^2$, $m^2 - 2mn + n^2$, and $2m^2 - 2n^2$.

7. $3x - 8y$, $7y - 6z$, and $5z - 2x$.

8. $2a^2 - 5ab - b^2$, $7a^2 + 3ab - 9b^2$, and $-4a^2 - 6ab + 8b^2$.

9. $4x - 3x^2 - 11 + 5x^3$, $12x^2 - 7 - 8x^3 - 15x$,
and $14 + 6x^3 + 10x - 9x^2$.

Note. It is convenient to arrange the first expression in *descending* powers of x (§ 33), as follows:

$$5x^3-3x^2+4x-11;$$

and then write the other expressions underneath the first, similar terms being in the same vertical column.

10. $2a-3b-5c$, $8b+6c+7d$, $-4a-3c+2d$, and $7a-b-9d$.

11. $x^3-3xy^2-2x^2y$, $3x^2y-5y^3-4xy^2$, $5xy^2-6y^3-7x^3$, and $8y^3+7x^3-9x^2y$.

12. $6a-8b-2c$, $12c+9d-7a$, $11b-10c-5d$, and $-3b-4d+a$.

13. $15a^3-2-9a^2-3a$, $13a-5a^2-6-7a^3$, $8+4a-8a^3-7a^2$, and $16a^2+3a^3-10a-2$.

14. $9a^2-13b^2-18c^2$, $15c^2+12b^2-8d^2$, $19d^2-14a^2+3c^2$, and $-2b^2-16d^2+11a^2$.

15. $12a^3-x^3+4ax^2-5a^2x$, $18x^3-2a^2x-3a^3-13ax^2$, $15a^2x-11x^3-17a^3+3ax^2$, and $6ax^2-8a^2x-7x^3+9a^3$.

16. $13x^2+3-4x+8x^3$, $-9x+5+16x^3+x^2$, $-15-6x^2-7x^3+11x$, and $-10x^3-12x+14x^2-17$.

SUBTRACTION.

35. Subtraction, in Algebra, is the process of finding one of two numbers, when their sum and the other number are given.

The *Minuend* is the sum of the numbers.

The *Subtrahend* is the given number.

The *Remainder* is the required number.

36. The remainder when b is subtracted from a is expressed $a - b$ (§ 3); and the remainder when $-b$ is subtracted from a is expressed $a - (-b)$.

37. Let it be required to subtract $-b$ from a.

By § 35, the sum of the remainder and the subtrahend is equal to the minuend.

Therefore, the required remainder must be such an expression that, when it is added to $-b$, the result shall equal a.

Now if $a + b$ be added to $-b$, the result is a.

Hence, the required remainder is $a + b$.

That is, $$a - (-b) = a + b.$$

38. From §§ 36 and 37, we have the following rule:

To subtract one number from another, change the sign of the subtrahend, and add the result to the minuend.

SUBTRACTION OF MONOMIALS.

39. **1.** Subtract $5a$ from $2a$.

It is convenient to place the subtrahend under the minuend.

We then change the sign of the subtrahend, giving $-5a$, and add the result to the minuend. Thus,

$$\begin{array}{r} 2a \\ \underline{-5a} \\ -3a, \textit{ Ans.} \end{array}$$

2. Subtract $-5a$ from $-2a$.

The student should perform *mentally* the operation of changing the sign of the subtrahend; thus, in Ex. 2, we mentally change $-5a$ to $5a$, and then add $5a$ to $-2a$.

$$\begin{array}{r} -2a \\ \underline{-5a} \\ 3a, \textit{ Ans.} \end{array}$$

EXAMPLES.

Subtract the following:

3. 7 from 4.

4. 4 from -11.

5. -15 from -9.

6. -9 from -25.

7. 18 from 5.

8. -26 from -18.

9. -5 from 16.

10. 12 from -17.

11. -14 from 13.

12.	**13.**	**14.**	**15.**	**16.**
$15a$	$-12x^2$	$-7ab$	$14m^3n$	$27xyz$
$6a$	$-31x^2$	$17ab$	$-8m^3n$	$34xyz$

17. $-xy$ from xy.

18. $-16a^3$ from $-44a^3$.

19. $21m^2n^2$ from $39m^2n^2$.

20. $19abc$ from $-6abc$.

21. $-45ax^4$ from $-19ax^4$.

22. $31a^2b^3$ from $8a^2b^3$.

23. From $8a$ take $-12b$.

24. From $-3m^2$ take $4n^2$.

25. From $-23a$ take the sum of $19a$ and $-5a$.

Note. A convenient way of performing examples like the above is to write the given expressions in a vertical column, *change the sign of each expression which is to be subtracted*, and then add the results.

26. From the sum of $-18xy$ and $11xy$, take the sum of $-29xy$ and $17xy$.

27. From the sum of $26a^2$ and $-7a^2$, take the sum of $-15a^2$ and $48a^2$.

28. From the sum of $33n^3x$ and $-16n^3x$, take the sum of $49n^3x$, $-27n^3x$, and $-39n^3x$.

SUBTRACTION OF POLYNOMIALS.

40. A polynomial may be regarded as the sum of its separate monomial terms (§ 28); hence,

To subtract one polynomial from another, change the sign of each term of the subtrahend, and add the result to the minuend.

1. Subtract $7ab^2 - 9a^2b + 8b^3$ from $5a^3 - 2a^2b + 4ab^2$.

It is convenient to place the subtrahend under the minuend so that similar terms shall be in the same vertical column.

We then *mentally* change the sign of each term of the subtrahend, and add the result to the minuend. Thus,

$$\begin{array}{rrrr}
5a^3 & -\,2a^2b & +\,4ab^2 & \\
 & -\,9a^2b & +\,7ab^2 & +\,8b^3 \\
\hline
5a^3 & +\,7a^2b & -\,3ab^2 & -\,8b^3,\ \textit{Ans.}
\end{array}$$

EXAMPLES.

Subtract the following:

2.
$$\begin{array}{r}
12a^2 - 9a - 7 \\
8a^2 - 6a + 13 \\
\hline
\end{array}$$

3.
$$\begin{array}{r}
2ab + 5bc - 3ca \\
-ab + 11bc - 4ca \\
\hline
\end{array}$$

4. From $x^2 - 2xy + y^2$ subtract $x^2 + 2xy + y^2$.

5. From $5a - 3b + 4c$ subtract $5a + 3b - 4c$.

6. From $4x^3 - 9x^2 + 11x - 18$ take $3x^3 - 8x^2 + 17x - 25$.

7. From $8x - 3y - 4z$ take $-z + 11x - 6y$.

8. Take $7b - 9c - 2d$ from $6a - 5b + 12c$.

9. Take $12a^3 + 4a - 9$ from $3a^3 + 8a^2 - 6$.

10. Subtract $x^3 - 7 - 2x - 6x^2$ from $5x^2 - 12 + 9x^3 - 2x$.

(See Note to Ex. 9, page 21.)

11. Subtract $1 + a^3 - a - a^2$ from $3a - 3a^2 + 1 - a^3$.

12. Take $81b^2 + 4a^2 - 36ab$ from $-30ab + 9a^2 + 25b^2$.

13. From $10x^3 - 21x^2 - 11x$ take $-15x^2 - 20x + 12$.

14. From $17a^3 - 12ab^2 + 5b^3$ take $8a^3 - 3a^2b + 13b^3$.

15. Take $-x^3 + 3x^2y - 3xy^2 + y^3$ from $x^3 - 2x^2y - 2xy^2 + y^3$.

16. Take $6c - 5d - 9b - 4a$ from $-10b - 2c + 3a - 9d$.

17. Subtract $4 - 3x - x^2 + 8x^3 + 10x^4$
from $9 - 7x + 6x^2 - 12x^3 + 5x^4$.

18. Subtract $2x^2 - xy + 8y^2 - 9x - 14y$ from $3x^2 - 5xy + 2y^2 - 2x + 7y$.

19. From $7a - 11a^3 - 8 + 6a^5$ subtract $16a^2 - 9 + 2a^5 + 15a - 10a^4$.

20. From $x^5 + 3x^4y - x^3y^2 + 5x^2y^3 - 4xy^4$ subtract $8x^4y - 7x^3y^2 - 6x^2y^3 + 11xy^4 - y^5$.

21. From $a^2 + 2ab + b^2$ subtract the sum of $-a^2 + 2ab - b^2$ and $-2a^2 + 2b^2$.

Note. Write the expressions one underneath the other, similar terms being in the same vertical column, change the signs of the terms of each expression which is to be subtracted, and add the results.

22. From the sum of $3a^2 + 2ab - b^2$ and $5a^2 - 8ab + 6b^2$, take $6a^2 - 5ab + 5b^2$.

23. Subtract the sum of $9x^2 - 8x + x^3$ and $5 - x^2 + x$ from $6x^3 - 7x - 4$.

24. Subtract the sum of $x + y - 8z$ and $-4x + 9y$ from the sum of $9x - 2y - z$ and $-5x + 6y - 7z$.

25. Take the sum of $6 - 4x^3 - x$ and $5x - 1 - 2x^2$ from the sum of $2x^3 + 7 - 4x - 5x^2$ and $3x^2 - 6x^3 - 2 + 8x$.

26. From the sum of $2a - 3b + 4d$ and $2b + 4c - 3d$, take the sum of $-4a - 4b + 3c - 2d$ and $3a - 2c$.

27. From the sum of $9a^3 - a^2 - 5$ and $3a^2 - a + 1$, take the sum of $-8a^3 + 13a + 3$ and $5a^3 + 2a^2 - 6a$.

IV. PARENTHESES.

REMOVAL OF PARENTHESES.

41. The expressions $a - b + (c - d)$

and $a - b - (c - d)$

indicate that the expression $c - d$ is to be respectively added to, and subtracted from, $a - b$.

If the operations be performed, we have by §§ 34 and 40,

$$a - b + (c - d) = a - b + c - d,$$

and

$$a - b - (c - d) = a - b - c + d.$$

In the first case, the signs of the terms within the parenthesis are *not changed* when the parenthesis is removed; while in the second case, the sign of each term within is *changed*, from + to −, or from − to +.

We then have the following rules:

A parenthesis preceded by a + sign may be removed without changing the signs of the terms enclosed.

A parenthesis preceded by a − sign may be removed if the sign of each term enclosed be changed, from + to −, or from − to +.

42. The above rules apply equally to the removal of the *brackets, braces,* or *vinculum* (§ 7).

It should be noticed in the case of the latter that the sign apparently prefixed to the first term underneath is in reality prefixed to the vinculum; thus, $+\overline{a - b}$ means the same as $+(a - b)$, and $-\overline{a - b}$ the same as $-(a - b)$.

43. 1. Remove the parentheses from

$$2a - 3b - (5a - 4b) + (4a - b).$$

By the rules of § 41, the expression becomes

$$2a - 3b - 5a + 4b + 4a - b = a, \textit{Ans.}$$

Parentheses are often found enclosing others; in this case they may be removed in succession by the rules of § 41; and it is better to remove first the *innermost* pair.

2. Simplify $4x - \{3x + (-2x - \overline{x - a})\}$.

Removing the vinculum first, and the others in succession, we have

$$\begin{aligned} &4x - \{3x + (-2x - \overline{x - a})\} \\ =&4x - \{3x + (-2x - x + a)\} \\ =&4x - \{3x - 2x - x + a\} \\ =&4x - 3x + 2x + x - a = 4x - a, \text{ Ans.} \end{aligned}$$

EXAMPLES.

Simplify the following expressions by removing the parentheses, etc., and uniting similar terms:

3. $8a + (5b - a) - (-7b + 2a)$.
4. $4m - [2m + 9n] - \{-5m - 6n\}$.
5. $x + y - z + \overline{y - z - x} - \overline{z - x + y}$.
6. $ab - 4b^2 - (2a^2 - b^2) - \{-5a^2 + 2ab - 3b^2\}$.
7. $m^2 - 3mn + \overline{5m^2 - mn - 6n^2} - [8m^2 - 4mn - 7n^2]$.
8. $4x - (5x - [3x - 1])$.
9. $a - (b - \overline{c + d} + e)$.
10. $5ab - [(3ab - 10) - (-4ab - 7)]$.
11. $7x^2 + (-3x^2 + \overline{2x - 5}) - (4x^2 - \overline{6x - 2})$.
12. $m - (6m - 7n) - \{-3m + 4n - (2m - 3n)\}$.
13. $17 - [45 - (9 - \overline{23 - 32})]$.
14. $3a - (5a - \{-7a + [9a - 4]\})$.
15. $x - [2x - (-x + 1) + 3] - \{6x - [-(x - 3) - x]\}$.
16. $x - (y + z - [x - (-x - y) + z]) + \{z - \overline{2x - y}\}$.
17. $2n - [3n - \{4n - \overline{n - 4}\} - (-5n - 9)]$.
18. $28 - \{-16 - (-4 + [55 - \overline{31 + 47}])\}$.

19. $a-(2a-[3a-\{4a-\overline{5a-1}\}])$.

20. $c-[2c-(6a-b)-\{c-\overline{5a+2b}-(-5a+\overline{6a-3b})\}]$

21. $x-[y-\{x-z-\overline{x-y+z}\}+(2x-\{-x+y\})]$.

22. $5x-[2x-(-x-\{2x-\overline{x-y}\}-3x)-3x]$.

23. $a-\{-a-[-a-(-a-\{-a-\overline{a-1}\})]\}$.

INTRODUCTION OF PARENTHESES.

44. To *enclose* any number of terms in a parenthesis, we take the converse of the rules of § 41:

Any number of terms may be enclosed in a parenthesis preceded by a + sign, without changing their signs.

Any number of terms may be enclosed in a parenthesis preceded by a − sign, if the sign of each term be changed, from + to −, or from − to +.

1. Enclose the last three terms of $a-b+c-d+e$ in a parenthesis preceded by a − sign.

Result, $a-b-(-c+d-e)$.

EXAMPLES.

In each of the following expressions, enclose the last three terms in a parenthesis preceded by a − sign:

2. $a+b-c-d$.

3. x^3-5x^2-8x+7.

4. $m^3+m^2n+mn^2+n^3$.

5. $a^2-b^2+2bc-c^2$.

6. $x^3+y^3+z^3-3xyz$.

7. $a-b-c+d+e$.

8. $a^4+6a^3+a^2-9a+2$.

9. $x^2-m^2-2mn-n^2$.

10. In each of the above results, enclose the last two terms in parenthesis in brackets preceded by a − sign.

V. MULTIPLICATION.

45. The Law of Signs.

If a and b are any two numbers, we have by § 19,

$$(+a)\times(+b)=+ab, \qquad (+a)\times(-b)=-ab,$$
$$(-a)\times(+b)=-ab, \qquad (-a)\times(-b)=+ab.$$

From these results, we may state the *Rule of Signs* in Multiplication as follows:

$+$ *multiplied by* $+$, *and* $-$ *multiplied by* $-$, *produce* $+$;

$+$ *multiplied by* $-$, *and* $-$ *multiplied by* $+$, *produce* $-$.

Or, as it is usually expressed with regard to the product of two terms,

Like signs produce $+$, *and unlike signs produce* $-$.

46. The Index Law.

Let it be required to multiply a^3 by a^2.

By § 6, $a^3=a\times a\times a,$

and $a^2=a\times a.$

Whence, $a^3\times a^2=a\times a\times a\times a\times a=a^5.$

Therefore, *the exponent of a letter in the product is equal to its exponent in the multiplicand plus its exponent in the multiplier.*

Or in general, if m and n are any two positive integers,

$$a^m\times a^n=a^{m+n}.$$

A similar result holds for the product of three or more powers of a.

Thus, $a^3\times a^4\times a^5=a^{3+4+5}=a^{12}.$

MULTIPLICATION OF MONOMIALS.

47. Let it be required to multiply $7a$ by $-2b$.

We have, $$-2b = (-2) \times b. \qquad (\S\ 45)$$

Whence, $$7a \times (-2b) = 7a \times (-2) \times b.$$

Then since the *order* of the factors is immaterial (§ 18),

$$\begin{aligned} 7a \times (-2b) &= 7 \times (-2) \times a \times b \\ &= -14ab. \qquad (\S\ 19) \end{aligned}$$

48. From §§ 45, 46, and 47, we derive the following rule for the multiplication of two monomials:

To the product of the absolute values of the numerical coefficients, annex the letters; giving to each an exponent equal to its exponent in the multiplicand plus its exponent in the multiplier.

Make the product + when the multiplicand and multiplier are of like sign, and − when they are of unlike sign.

1. Multiply $2a^5$ by $9a^4$.

By the rule, $2a^5 \times 9a^4 = 2 \times 9 \times a^{5+4} = 18a^9$, *Ans.*

2. Multiply a^3b^2c by $-5a^2bd$.

We have, $a^3b^2c \times (-5a^2bd) = -5a^5b^3cd$, *Ans.*

3. Multiply $-7x^m$ by $4x^3$.

We have, $(-7x^m) \times 4x^3 = -28x^{m+3}$, *Ans.*

4. Multiply $-3x^n$ by $-8x^n$.

We have, $(-3x^n) \times (-8x^n) = 24x^{n+n} = 24x^{2n}$, *Ans.*

EXAMPLES.

Multiply the following:

5. $7a^3$ by $3a^7$.	7. $5xyz$ by $-11xyz$.
6. $-14ab$ by $2cd$.	8. $-15a^6b$ by $-4ab^5$.

9.	$-9m^2n^3$ by $7m^2n^3$.	**13.**	$-a^mb^nc^p$ by $-ab^3c^2$.
10.	$-6a^3b^6$ by $-b^3c^5$.	**14.**	$-8x^my^n$ by $12x^my^n$.
11.	$8x^2z^7$ by $-8y^3z^3$.	**15.**	$10a^4b^5c^2$ by $9a^7c^6d^3$.
12.	$12a^2bc$ by $6bcd^2$.	**16.**	$16x^{3p}y^q$ by $-8x^{2p}y^r$.

49. We have by § 45,

$$(-a)\times(-b)\times(-c)=(ab)\times(-c)=-abc; \qquad (1)$$

$$(-a)\times(-b)\times(-c)\times(-d)=(-abc)\times(-d), \text{ by (1)},$$
$$=abcd; \text{ etc.}$$

That is, the product of three negative terms is negative; the product of four negative terms is positive; and so on.

Hence, *the product of any number of terms is positive or negative according as the number of negative terms is even or odd.*

1. Required the product of $-2a^2b^3$, $6bc^5$, and $-7c^2d$.

Since there are two negative terms, the product is positive.

Whence, $(-2a^2b^3)\times(6bc^5)\times(-7c^2d)=84a^2b^4c^7d$, *Ans.*

EXAMPLES.

Multiply the following:

2. $3a^5$, $5a^6$, and $-6a^4$.
3. $-4x^4$, $-9y^2$, and $2z^3$.
4. $x^{2m}y^n$, y^nz^p, and x^5z^q.
5. $-12a^7b^2$, $-b^4c^3$, and $-8c^5a^4$.
6. a^5, $3a$, $5a^7$, and $-7a^3$.
7. $-2a^3b^5c$, $2a^5bd^3$, $-2ac^3d^5$, and $2b^3c^5d$.
8. $2x^3y^4$, $-3yz^4$, $-4z^3x^4$, and $-5x^6y^2z^3$.
9. $-a^mx^p$, $-b^{2n}y$, $-a^{3m}y^q$, and $-b^nx$.
10. $5ab^4$, $-4a^5c^7$, $-a^6d^3$, $6b^2c$, and $-3c^2d^5$.

MULTIPLICATION OF POLYNOMIALS BY MONOMIALS.

50. In § 30, we showed that the product of $a + b$ and c was $ac + bc$.

We then have the following rule for the product of a polynomial by a monomial:

Multiply each term of the polynomial by the monomial, and unite the results with their proper signs.

1. Multiply $-8x^3$ by $2x^2 - 5x + 7$.

Multiplying each term of $2x^2 - 5x + 7$ by $-8x^3$, we have

$$(2x^2 - 5x + 7) \times (-8x^3) = -16x^5 + 40x^4 - 56x^3, \textit{ Ans.}$$

EXAMPLES.

Multiply the following:

2. $4a - 9$ by $5a$.
3. $8x^2y - 5xy^3$ by $-3xy^4$.
4. $a^2 - ab + b^2$ by ab.
5. $3x^2 + x - 8$ by $-9x^2$.
6. $-7a^3$ by $2a^3 - 6a^2 - 7$.
7. m^3n^3 by $m^2 - 2mn + n^2$.
8. $8a^pb^n - 9ab^2$ by $-6a^qb^3$.
9. $6x^5 - 5x^6 - 7x^4$ by $-8x^6$.

(See note to Ex. 9, p. 21.)

10. $-4b^2 - a^2 + 5ab$ by $4a^2b^3$.
11. $-x^2y^2$ by $x^3 - 3x^2y + 3xy^2 - y^3$.
12. $5a^3 + 9 - 8a^4 - 4a - a^2$ by $7a^5$.
13. $-2mn$ by $3m^3 - 6m^2n - 7mn^2 + 2n^3$.

MULTIPLICATION OF POLYNOMIALS BY POLYNOMIALS.

51. Let it be required to multiply $a + b$ by $c + d$.

As in § 30, we multiply $a + b$ by c, and then $a + b$ by d, and add the second result to the first.

That is,
$$\begin{aligned}(a + b)(c + d) &= (a + b)c + (a + b)d \\ &= ac + bc + ad + bd.\end{aligned}$$

We then have the following rule:

Multiply the multiplicand by each term of the multiplier, and add the partial products.

52. 1. Multiply $3a-4b$ by $2a-5b$.

In accordance with the rule, we multiply $3a-4b$ by $2a$, and then by $-5b$, and add the partial products.

A convenient arrangement of the work is shown below, similar terms being in the same vertical column.

$$\begin{array}{lll} 3a & -4b & \\ 2a & -5b & \\ \hline 6a^2 & -\ 8ab & \\ & -15ab & +20b^2 \\ \hline 6a^2 & -23ab & +20b^2, \textit{ Ans.} \end{array}$$

Note. The work may be *verified* by performing the example with the multiplicand and multiplier interchanged.

2. Multiply $4ax^2+a^3-8x^3-2a^2x$ by $2x+a$.

It is convenient to arrange the multiplicand and multiplier in the same order of powers of some common letter (§ 33), and write the partial products in the same order.

Arranging the expressions according to the descending powers of a, we have

$$\begin{array}{llllll} a^3 & -2a^2x & +4ax^2 & -8x^3 & & \\ a & +2x & & & & \\ \hline a^4 & -2a^3x & +4a^2x^2 & -8ax^3 & & \\ & \ \ 2a^3x & -4a^2x^2 & +8ax^3 & -16x^4 & \\ \hline a^4 & & & & -16x^4, & \textit{Ans.} \end{array}$$

EXAMPLES.

Multiply the following:

3. $2a+5$ by $3a+7$.

4. $5a-8$ by $6a-1$.

5. $-4x-5y$ by $8x+3y$.

6. $-7ab+2$ by $-4ab-6$.

7. x^2-xy+y^2 by $x+y$.

8. $2a^2+7a-9$ by $5a-1$.

9. $3x^2 - x + 4$ by $4x - 3$.

10. $-8n + 5n^2 - 3$ by $2 + n$.

11. $3a - 2b$ by $9a^2 + 6ab + 4b^2$.

12. $a - b + c$ by $a - b + c$.

13. $6m^2 - 5mn - 8n^2$ by $2m^3 + 3m^2n$.

14. $x^3 + 3x^2 - 7x - 6$ by $3x - 4$.

15. $m^2 + mn + n^2$ by $m^2 - mn + n^2$.

16. $8a^3 - 4a^2 + 2a - 1$ by $2a + 1$.

17. $9x^2 - 5 + 6x$ by $8x + 4 + 7x^2$.

18. $6n - 8 + 4n^2$ by $-4 + 2n^2 - 3n$.

19. $3a^2 - 5ab - 8b^2$ by $4a^2 - 9ab - 7b^2$.

20. $2x + 6z - 4y$ by $2y - 3z + x$.

21. $4a + 6b + 10c$ by $2a - 3b + 5c$.

22. $a^3 - 2a^2 + a - 2$ by $a^2 + 2a + 3$.

23. $x^4 + 2x^3 + 4x^2 + 8x + 16$ by $x - 2$.

24. $m^3 + n^3 + mn^2 + m^2n$ by $m^2n - mn^2$.

25. $-5x^2 + 9 + 2x^3 - 4x$ by $5x^2 - 1 + 6x$.

26. $4x^{2m}y - 5x^3y^{2n-3}$ by $2x^{2m-1}y^{n+4} - 3x^2y^{3n}$.

27. $3m^3 - 5m^2 + 4m - 1$ by $2m^2 - m - 3$.

28. $16a^4 - 24a^3 + 36a^2 - 54a + 81$ by $2a + 3$.

29. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^2 - 2ab + b^2$.

30. $x^3 - 2x^2 - x - 1$ by $x^3 + 2x^2 - x + 1$.

31. $a^3 - 6a^2 + 12a - 8$ by $a^3 + 6a^2 + 12a + 8$.

32. $m^2 - 6mn - 7n^2$ by $m^3 - 2m^2n - 5mn^2 + 4n^3$.

33. $x^3 - 3 + 2x^2 - x$ by $3 - x + x^3 - 2x^2$.

34. $a^2 + b^2 + c^2 + ab - bc + ca$ by $a - b - c$.

35. $6x^3 - 4x^2y - 3xy^2 + 2y^3$ by $2x^2 + xy - 2y^2$.

Find the product of the following:

36. $x-2$, $x-3$, and $x-4$.

37. $a+5$, $2a-3$, and $4a-1$.

38. $x-y$, x^2+xy+y^2, and x^3+y^3

39. $3n-8$, $4n+7$, and $5n-6$.

40. $a-x$, $a+x$, a^2+x^2, and a^4+x^4.

41. $m-4n$, $2m+3n$, and $2m^2+5mn-12n^2$.

42. $a+1$, $a-1$, a^2+a+1, and a^2-a+1.

43. x^2+x+1, x^2-x+1, and x^4-x^2+1.

44. $a+b$, $a-b$, $2a-3b$, and $2a+3b$.

45. $x+3$, $2x+1$, $2x-1$, and $4x^3-12x^2+x-3$.

53. 1. Simplify $(a-2x)^2-2(x+3a)(a-x)$.

To simplify the expression, we should first multiply $a-2x$ by itself (§ 6); we should then find the product of 2, $x+3a$, and $a-x$, and subtract the second result from the first.

$$\begin{array}{lll} a & -2x & \\ a & -2x & \\ \hline a^2 & -2ax & \\ & -2ax & +4x^2 \\ \hline a^2 & -4ax & +4x^2 \end{array} \qquad\qquad \begin{array}{rrr} 3a & +x & \\ a & -x & \\ \hline 3a^2 & +ax & \\ & -3ax & -x^2 \\ \hline 3a^2 & -2ax & -x^2 \\ & & 2 \\ \hline 6a^2 & -4ax & -2x^2 \end{array}$$

Subtracting the second result from the first, we have

$$a^2-4ax+4x^2-6a^2+4ax+2x^2=-5a^2+6x^2, \textit{ Ans.}$$

EXAMPLES.

Simplify the following:

2. $(3x-8)(x+6)+(2x-7)(4x+9)$.

3. $(2a+5)(3a-7)-(2a-5)(3a+7)$.

4. $(a - m)(b + n) + (a + m)(b - n)$.
5. $(x - y + z)^2 - (x + y - z)^2$.
6. $(a - b - c + d)^2$.
7. $(2x + 3)^2(2x - 3)^2$.
8. $(a + b)(a^2 - b^2) - (a - b)(a^2 + b^2)$.
9. $(3x - 5y)^2 - 5(x - y)(x - 5y)$.
10. $(a + x)(a^3 + x^3)[a^2 - x(a - x)]$.
11. $(a - b)(a^3 + a^2b + ab^2 + b^3)[(a^2 + b^2)^2 - 2a^2b^2]$.
12. $(x + 1)(x + 2)(x + 3) - (x - 1)(x - 2)(x - 3)$.
13. $(x - y)(y - z) - (x - z)(y - z) - (x - y)(x - z)$.
14. $(a + b + c)^2 - (a + b)^2 - c(2a + 2b + c)$.
15. $(a + 1)(2a + 5)(4a - 3) + (a - 1)(2a - 5)(4a + 3)$.
16. $(x + y - z)^2 + (y + z - x)^2 + (z + x - y)^2$.
17. $2(a + 2x)(a - 2x)[(a + 2x)^2 + (a - 2x)^2]$.
18. $(a + b + c)^2 - (a + b - c)^2 - (a - b + c)^2 + (a - b - c)^2$.
19. $[(m + n)^2 + (m - n)^2][(2m + n)^2 - (m - 2n)^2]$.
20. $(a + b - c)(b + c - a)(c + a - b)$.
21. $(a + b)^3 - (a - b)^3$.
22. $(x + y + z)^3 - 3(x^2 + y^2 + z^2)(x + y + z)$.

VI. DIVISION.

54. Division, in Algebra, is the process of finding one of two numbers, when their product and the other number are given.

The *Dividend* is the product of the numbers.

The *Divisor* is the given number.

The *Quotient* is the required number.

55. The Law of Signs.

Since the dividend is the product of the divisor and quotient, the equations of § 45 may be written as follows:

$$(+ab) \div (+a) = +b, \qquad (-ab) \div (+a) = -b,$$

$$(-ab) \div (-a) = +b, \qquad (+ab) \div (-a) = -b.$$

From these results, we may state the *Rule of Signs* in Division as follows:

+ *divided by* +, *and* − *divided by* −, *produce* +;

+ *divided by* −, *and* − *divided by* +, *produce* −.

Hence, in Division as in Multiplication,

Like signs produce +, *and unlike signs produce* −.

56. The Index Law.

Let it be required to divide a^5 by a^2.

The quotient must be a number which, when multiplied by the divisor, a^2, will produce the dividend, a^5.

Now if a^3 be multiplied by a^2, the product is a^5.

Whence, $$\frac{a^5}{a^2} = a^3.$$

Hence, *the exponent of a letter in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.*

DIVISION OF MONOMIALS.

57. Let it be required to divide $-14\,a^2b$ by $7\,a^2$.

We find a number which, when multiplied by $7\,a^2$, will produce $-14\,a^2b$.

That number is evidently $-2\,b$.

Whence, $$\frac{-14\,a^2b}{7\,a^2} = -2\,b.$$

58. From §§ 55, 56, and 57, we derive the following rule for the division of two monomials:

To the quotient of the absolute values of the numerical coefficients, annex the letters; giving to each an exponent equal to its exponent in the dividend minus its exponent in the divisor, and omitting any letter having the same exponent in the dividend and divisor.

Make the quotient + when the dividend and divisor are of like sign, and − when they are of unlike sign.

1. Divide $54\,a^7$ by $-9\,a^4$.

By the rule, $$\frac{54\,a^7}{-9\,a^4} = -6\,a^{7-4} = -6\,a^3, \textit{ Ans.}$$

2. Divide $-2\,a^3b^2cd^4$ by abd^4.

We have, $$\frac{-2\,a^3b^2cd^4}{abd^4} = -2\,a^2bc, \textit{ Ans.}$$

3. Divide $-91\,x^{2m}y^nz^r$ by $-13\,x^my^nz^3$.

We have, $$\frac{-91\,x^{2m}y^nz^r}{-13\,x^my^nz^3} = 7\,x^{2m-m}z^{r-3} = 7\,x^mz^{r-3}, \textit{ Ans.}$$

EXAMPLES.

Divide the following:

4. 35 by -5.

5. -44 by 11.

6. -64 by -4.

7. -84 by 7.

8. -144 by -8.

9. 168 by -12.

10. $16a^7$ by $4a^4$.

11. $-18x^3y$ by $2xy$.

12. $2m^5n^4$ by $-m^2n^3$.

13. $-a^6b^3c^2$ by $-a^4b^3c$.

14. $-6x^7y^{11}$ by $6x^3y^6$.

15. $-24a^4b^2$ by $-8a^4b^2$.

16. $28x^5yz^3$ by $-7x^3z^2$.

17. $40m^4n^7$ by $5m^2n$.

18. $-33a^6x^2y^4$ by $-3a^5y$.

19. $-36a^{3m+1}$ by $12a^{2m-3}$.

20. $81a^4b^9c^5$ by $9b^2c^4$.

21. $65x^ny^5z^3$ by $-13xy^5$.

22. $-a^pb^r$ by $-a^qb^s$.

23. $54x^{3m}y^{4n}$ by $9x^my^n$.

24. $98a^9b^7c^8$ by $-14a^2bc^7$.

25. $-143m^{13}n^6p^{12}$ by $11m^5n^6p^3$.

DIVISION OF POLYNOMIALS BY MONOMIALS.

59. We have, $a(b+c) = ab + ac$.

Since the dividend is the product of the divisor and quotient (§ 54), we may regard $ab + ac$ as the dividend, a as the divisor, and $b + c$ as the quotient.

Whence,
$$\frac{ab + ac}{a} = b + c.$$

We then have the following rule:

Divide each term of the dividend by the divisor, and unite the results with their proper signs.

1. Divide $9a^3b^2 - 6a^4c + 12a^2bc^3$ by $-3a^2$.

$$\frac{9a^3b^2 - 6a^4c + 12a^2bc^3}{-3a^2} = -3ab^2 + 2a^2c - 4bc^3, \textit{ Ans.}$$

EXAMPLES.

Divide the following:

2. $16x^9 + 28x^6 - 24x^3$ by $4x^3$.

3. $104mn^3 - 39m^3n$ by $-13mn$.

4. $6a^2b^7c^3 - 15a^6b^3c^5 + 3a^4b^8c$ by $-3a^2b^3$.

5. $-63x^5y^6z^2 - 84x^3y^4z^7$ by $7x^3yz^2$.

6. $20m^8n^5 - 45m^6n^7 - 35m^4n^9$ by $-5m^4n^5$.

7. $-24a^{11} + 108a^9 + 84a^7$ by $12a^6$.

8. $40a^3bc - 24ab^3c - 32abc^3$ by $-8abc$.

9. $72x^{10} - 9x^8 + 54x^6 - 99x^4$ by $-9x^4$.

10. $-2x^4y + 6x^3y^2 - 6x^2y^3 + 2xy^4$ by $-2xy$.

11. $60a^{14} - 30a^{12} + 15a^{10} - 45a^8$ by $15a^7$.

12. $a^{3m}b^n - 3a^{2m}b^{2n} + 2a^mb^{3n}$ by a^mb^n.

13. $48a^7b^8c^4 + 36a^3b^6c^8 - 30a^6b^5c^9$ by $6a^3b^5c^4$.

14. $-88x^4y^8z^2 + 55xy^9z^4 + 66x^3y^6z^5$ by $-11xy^6z^2$.

15. $x^{n+2}y^{q+1}z^3 - x^ny^{2q}z^4 - x^my^qz^r$ by $-x^ny^qz^3$.

DIVISION OF POLYNOMIALS BY POLYNOMIALS.

60. Let it be required to divide $12 + 10x^3 - 11x - 21x^2$ by $2x^2 - 4 - 3x$.

Arranging each expression according to the descending powers of x (§ 33), we are to find an expression which, when multiplied by the divisor, $2x^2 - 3x - 4$, will produce the dividend, $10x^3 - 21x^2 - 11x + 12$.

It is evident that the term containing the highest power of x in the product is the product of the terms containing the highest powers of x in the multiplicand and multiplier.

Therefore, $10x^3$ is the product of $2x^2$ and the term containing the highest power of x in the quotient.

Whence, the term containing the highest power of x in the quotient is $10x^3$ divided by $2x^2$, or $5x$.

Multiplying the divisor by $5x$, we have the product $10x^3 - 15x^2 - 20x$; which, when subtracted from the dividend, leaves the remainder $-6x^2 + 9x + 12$.

This remainder must be the product of the divisor by the rest of the quotient; therefore, to obtain the next term of the quotient, we regard $-6x^2 + 9x + 12$ as a new dividend.

Dividing the term containing the highest power of x, $-6x^2$, by the term containing the highest power of x in the divisor, $2x^2$, we obtain -3 as the second term of the quotient.

Multiplying the divisor by -3, we have the product $-6x^2+9x+12$; which, when subtracted from the second dividend, leaves no remainder.

Hence, $5x-3$ is the required quotient.

It is customary to arrange the work as follows:

$$\begin{array}{r|l}
10x^3-21x^2-11x+12 & 2x^2-3x-4, \text{ Divisor.} \\
\underline{10x^3-15x^2-20x} & 5x-3, \text{ Quotient.} \\
-\ 6x^2+\ 9x+12 & \\
\underline{-\ 6x^2+\ 9x+12} &
\end{array}$$

Note. The example might have been solved by arranging the dividend and divisor according to the *ascending* powers of x.

From the above example, we derive the following rule:

Arrange the dividend and divisor in the same order of powers of some common letter.

Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.

Multiply the whole divisor by the first term of the quotient, and subtract the product from the dividend.

If there be a remainder, regard it as a new dividend, and proceed as before; arranging the remainder in the same order of powers as the dividend and divisor.

61. 1. Divide $9ab^2+a^3-9b^3-5a^2b$ by $3b^2+a^2-2ab$.

Arranging according to the descending powers of a,

$$\begin{array}{r|l}
a^3-5a^2b+9ab^2-9b^3 & a^2-2ab+3b^2 \\
\underline{a^3-2a^2b+3ab^2} & a-3b, \textit{ Ans.} \\
-3a^2b+6ab^2 & \\
\underline{-3a^2b+6ab^2-9b^3} &
\end{array}$$

Note 1. In the above example, the last term of the second dividend is omitted, as it is merely a repetition of the term directly above.

Note 2. The work may be verified by multiplying the quotient by the divisor, which should of course give the dividend.

2. Divide $8 + 18x^4 - 56x^2$ by $-6x^2 + 4 + 8x$.

Arranging according to the ascending powers of x,

$$\begin{array}{l|lllll}
4 + 8x - 6x^2) & 8 & -56x^2 & +18x^4 & & (2 - 4x - 3x^2, \textit{ Ans.} \\
 & 8 + 16x & -12x^2 & & & \\ \hline
 & -16x & -44x^2 & +18x^4 & & \\
 & -16x & -32x^2 & +24x^3 & & \\ \hline
 & & -12x^2 & -24x^3 & +18x^4 & \\
 & & -12x^2 & -24x^3 & +18x^4 & \\ \hline
\end{array}$$

EXAMPLES.

Divide the following:

3. $15x^2 - 11x - 14$ by $3x + 2$.
4. $25m^2 + 40mn + 16n^2$ by $5m + 4n$.
5. $12a^2 - 28a + 15$ by $6a - 5$.
6. $x^3 - 6x^2 - 19x + 84$ by $x - 7$.
7. $8m^3 + 27n^3$ by $2m + 3n$.
8. $x^3 - 64y^3$ by $x - 4y$.
9. $8 - 16a + 6a^2$ by $3a - 2$.
10. $50x^2y^2 - 18$ by $3 - 5xy$.
11. $10a^4b^2 - 18a^2b^4 - 3a^3b^3$ by $2a^2b - 3ab^2$.
12. $2m^4 - 8m^3n + 18mn^3$ by $2m^2 - 6mn$.
13. $20 + 36a^3 - 49a$ by $12a^2 + 5 - 16a$.
14. $2a^5b^2 - 3a^4b^3 - 7a^3b^4 + 4a^2b^5$ by $a^3b - a^2b^2 - 4ab^3$.
15. $a^2 - b^2 + 2bc - c^2$ by $a + b - c$.
16. $4y^3 - 16x^2y + 6xy^2 + 6x^3$ by $3x^2 - y^2 - 2xy$.
17. $39mn^2 + 30m^3 - 20n^3 - 43m^2n$ by $6m - 5n$.

18. $4a^4-9a^2+30a-25$ by $2a^2+3a-5$.

19. $4x+x^4+3$ by $3+x^2-2x$.

20. n^4-16 by $2n^2+8+4n+n^3$.

21. $6m^4-19m^3+22m+5$ by $3m-5$.

22. $x^4+y^4+x^2y^2$ by y^2+x^2-xy.

23. $1-16a^8$ by $1+2a^2$.

24. $16x^4-81y^4$ by $2x-3y$.

25. $-9m^2-16+m^4-24m$ by $3m+m^2+4$.

26. $9x^4+4-13x^2$ by $3x^2-2+x$.

27. $2a^4-a^3+8a-5$ by $2a^2-3a+5$.

28. $13x^3+71x-70x^2-20+6x^4$ by $4+3x^2-7x$.

29. $4m^2n^4+n^8+16m^4$ by $2mn^2+4m^2+n^4$.

30. x^5+32 by $x+2$.

31. $120a^4+26a^3-111a^2-14a+24$ by $(3a+2)(4a-3)$.

32. $(2m^2-m-1)(3m^2+m-2)$ by $(2m+1)(3m-2)$.

33. a^5+243 by $9a^2+81-3a^3-27a+a^4$.

34. $4x^{2m+7}y^3-16x^{m+6}y^{n+1}+12x^5y^{2n-1}$ by $x^{m+2}y-3xy^{n-1}$.

35. $6a^5-6ab^4$ by $-3b+3a$.

36. $a^5-a^4b-ab^4+b^5$ by $a^2-2ab+b^2$.

37. $8m^5-14m^2-18m+21$ by $4m^3+6m-7$.

38. $16a^4-96a^3+216a^2-216a+81$ by $(2a-3)^2$.

39. $7x^3-6x^5-28+81x^2+3x-25x^4$ by $4-3x^2-5x$.

40. $2x^6-6x^5-x^4-9x^2+3x-9$ by $2x^3-x+3$.

41. $70a-50-a^5-37a^2$ by $6a-5-a^3-2a^2$.

42. $x^5-81xy^4+243y^5-3x^4y$ by $9xy^3+x^3+27y^3+3x^2y$.

43. $14x^4-23x+6x^6+6x^2-11x^5+5-12x^3$
by $5x-3x^2+2x^3-1$.

44. $4a^6-49a^4+76a^2-16$ by $2a^3+5a^2-6a-4$.

45. $m^6-6m^4n^2+9m^2n^4-4n^6$ by $(m+n)(m-n)(m+2n)$.

46. $8a^2-10ab+18ac-3b^2+8bc-5c^2$ by $2a-3b+5c$.

47. $x^{2p}-y^{2q}+2y^qz^r-z^{2r}$ by $x^p-y^q+z^r$.

The operation of division may be abridged in certain cases by the use of parentheses.

48. Divide $x^3+(a-b+c)x^2+(-ab-bc+ca)x-abc$
by $x+a$.

$$\begin{array}{l|l}
x^3+(a-b+c)x^2+(-ab-bc+ca)x-abc & x+a \\ \cline{2-2}
\underline{x^3 \qquad\quad +ax^2} \qquad\qquad\qquad\qquad\qquad & x^2+(-b+c)x-bc,\ \textit{Ans.} \\
\qquad (-b+c)x^2 & \\
\qquad \underline{(-b+c)x^2+(-ab \qquad +ca)x} & \\
\qquad\qquad\qquad\qquad\qquad -bcx & \\
\qquad\qquad\qquad\qquad\qquad \underline{-bcx-abc} &
\end{array}$$

Divide the following:

49. $x^3+(-a+b-c)x^2+(-ab-bc+ca)x+abc$
by $x^2+(-a+b)x-ab$.

50. $x^3+(a+b+c)x^2+(ab+bc+ca)x+abc$ by $x+c$.

51. $x^3+(3a-2b-c)x^2+(-6ab+2bc-3ac)x+6abc$
by $x^2+(3a-c)x-3ac$.

52. $a(a+b)x^2+(ab+b^2+bc)x-c(b+c)$ by $ax+(b+c)$.

53. $m(m-n)x^2+(-mn+n^2-np)x+p(n-p)$
by $mx-(n-p)$.

54. $x^3+(a-b-c)x^2+(-ab+bc-ca)x+abc$
by $x^2-(b+c)x+bc$.

55. $x^3-(a+b+c)x^2+(ab+bc+ca)x-abc$ by $x-a$.

56. $a^2(b-c)d+a(-b^2+c^2+d^2)-(b+c)d$
by $ad-(b+c)$.

57. $a^2+(m+n)a-2m^2+11mn-12n^2$ by $a-m+4n$.

EXAMPLES FOR REVIEW.

62. **1.** Find the numerical value when $a = 4$, $b = -7$, $c = -3$, and $d = 5$, of

$$(a+b)^2 - \frac{c-d}{c+d}.$$

We have, $(a+b)^2 = (4-7)(4-7) = (-3)(-3) = 9,$

and $\frac{c-d}{c+d} = \frac{-3-5}{-3+5} = \frac{-8}{2} = -4.$

Then, $(a+b)^2 - \frac{c-d}{c+d} = 9 - (-4) = 9 + 4 = 13$, *Ans.*

Find the numerical value of each of the following when $a = 5$, $b = -4$, $c = -2$, and $d = 3$:

2. $(a-b)(b+c)(c-d)$. **3.** $b^2 - c^2 + 2cd - d^2$.

4. $(a+3b)(4c-d) + (a-c)(2b+d)$.

5. $a^3 - 3a^2b + 3ab^2 - b^3$. **8.** $3a^2b - 5b^3c + 4c^4d$.

6. $\frac{8ad}{bc} - \frac{6ab}{cd}$. **9.** $(a-b)^3 - (c-d)^3$.

7. $\frac{a+2b}{4c+d} + \frac{a-5b}{6c-d}$. **10.** $\frac{26a+23b+64c}{11a+24b-7c}$.

11. $\frac{2a-b}{b-c} - \frac{3b-c}{c-d} + \frac{4c-d}{d-a}$.

12. Add $9(a-b) - 8(b-c)$, $-3(b-c) - 7(c-d)$, and $4(c-d) - 5(a-b)$.

$$\begin{array}{rrr}
9(a-b) & -\ 8(b-c) & \\
 & -\ 3(b-c) & -\ 7(c-d) \\
-5(a-b) & & +\ 4(c-d) \\
\hline
4(a-b) & -\ 11(b-c) & -\ 3(c-d),\ \textit{Ans.}
\end{array}$$

13. Add $4a^2(a+x) - 6(b-y)$, $-3a^2(a+x) - 2(b-y)$, and $-7a^2(a+x) + 8(b-y)$.

14. Add $18(x-y)^2 - 11(x+y)^3$, $-9(x-y)^2 + 7(x+y)^3$, and $-4(x-y)^2 - 5(x+y)^3$.

15. Subtract $5(a-b)-8(c+d)$ from $2(a-b)-3(c+d)$.

16. Multiply $3(x+y)-5$ by $3(x+y)+5$.

17. Multiply $7(a-b)+4$ by $9(a-b)-8$.

18. Divide $6(m+n)^2-(m+n)-15$ by $3(m+n)-5$.

19. Divide $(x-y)^3+1$ by $(x-y)+1$.

20. Add $\frac{3}{4}a+\frac{2}{5}b-\frac{1}{3}c$ and $\frac{1}{6}a-\frac{4}{3}b+\frac{5}{7}c$.

$$\begin{array}{r}
\frac{3}{4}a+\ \frac{2}{5}b-\ \frac{1}{3}c\\
\frac{1}{6}a-\ \frac{4}{3}b+\ \frac{5}{7}c\\
\hline
\frac{11}{12}a-\frac{14}{15}b+\frac{8}{21}c,\ \textit{Ans.}
\end{array}$$

21. Add $\frac{4}{3}a-\frac{3}{4}b+\frac{5}{2}c$ and $\frac{1}{2}a+\frac{2}{3}b-\frac{3}{5}c$.

22. Add $\frac{2}{5}x-\frac{1}{7}y-\frac{2}{15}z$ and $-\frac{5}{4}x+\frac{3}{2}y-\frac{1}{6}z$.

23. From $\frac{1}{3}a-\frac{3}{2}b+\frac{5}{4}c$ take $\frac{1}{2}a-\frac{6}{7}b-\frac{3}{8}c$.

24. Subtract $-\frac{3}{10}x+\frac{8}{9}y+\frac{1}{2}z$ from $-\frac{5}{6}x+\frac{2}{3}y-\frac{4}{5}z$.

25. Multiply $\frac{4}{9}x^2+\frac{1}{2}x+\frac{9}{16}$ by $\frac{2}{3}x-\frac{3}{4}$.

26. Multiply $\frac{1}{9}a^2-\frac{1}{6}ab+\frac{1}{16}b^2$ by $\frac{1}{3}a-\frac{1}{4}b$.

27. Divide $\frac{27}{8}x^3+\frac{8}{125}$ by $\frac{3}{2}x+\frac{2}{5}$.

28. Divide $\frac{3}{4}a^3-\frac{7}{8}a^2b+\frac{73}{72}ab^2-\frac{1}{3}b^3$ by $\frac{1}{2}a-\frac{4}{3}b$.

29. Multiply $a^{2p+3}b^4-a^6b^{3q+2}$ by $a^{2p-3}-b^{3q-2}$.

30. Divide $x^{2m-1}-x^3y^{4n+2}$ by $x^{m-2}+y^{2n+1}$.

31. Divide $a^{3p+4}-ab^{6q-3}$ by $a^{p+1}-b^{2q-1}$.

32. Add $3(x+1)^2-2(x+1)$, $5(x+1)-7$,
and $-(x+1)^2-3(x+1)+4$.

33. From $7(x+y)^2-9x(x+y)+4$
take $12(x+y)^2+x(x+y)-11$.

34. Simplify $5x-[3x-\{x-(7x-\overline{8x-4})\}-(9x-\overline{5x-2})]$

35. Add $\frac{5}{12}x^2-\frac{4}{9}x-\frac{7}{10}$, $-\frac{11}{24}x^2+\frac{7}{18}x-\frac{3}{4}$,
and $\frac{7}{8}x^2-\frac{5}{6}x+\frac{8}{15}$.

36. Multiply $x^2+(b-c)x-bc$ by $x+a$.

37. Divide $a^{2m+8}b^3 - a^2b^{2n-5}$ by $a^{m+3} - b^{n-4}$.

38. Subtract $\frac{7}{15}a^2 - \frac{1}{6}a + \frac{7}{10}$ from $\frac{3}{10}a^2 + \frac{5}{14}a - \frac{8}{35}$.

39. Multiply $(m-n)^2 + 2(m-n) + 1$
by $(m-n)^2 - 2(m-n) + 1$.

40. Multiply $a^{2n} - a^nb^n + b^{2n}$ by $a^{n+1}b^2 + ab^{n+2}$.

41. Simplify $(a+b)^2 - 2(a+b)(a-b) + (a-b)^2$.

42. Simplify $a-[2a-(b-6c)-\{a-(-2b-5c)-\overline{3b-c}\}]$

43. Multiply $\frac{3}{8}a^2 - \frac{1}{4}a - \frac{2}{3}$ by $\frac{3}{2}a^2 - a - \frac{8}{3}$.

44. Divide $\frac{9}{8}a^4 - \frac{3}{2}a^3 + \frac{1}{2}a^2 - \frac{2}{9}$ by $\frac{3}{2}a^2 - a - \frac{2}{3}$.

45. Divide $a^5 - b^5 - 5ab(a^3 - b^3) + 10a^2b^2(a-b)$
by $(a+b)^2 - 4ab$.

46. Divide $12x^{5m+1}y^{n-2} - 13x^{3m+4}y^{3n-4} - 35x^{m+7}y^{5n-6}$
by $4x^{3m+2}y^{n-5} + 5x^{m+5}y^{3n-7}$.

47. Multiply $(a+b)x - 2ab$ by $x + (a+b)$.

48. Divide $(a-b)^3 - 3(a-b)^2c + 3(a-b)c^2 - c^3$
by $(a-b) - c$.

49. Divide $x^{4m} + x^{2m}y^{2n} + y^{4n}$ by $x^{2m} + x^my^n + y^{2n}$.

50. Multiply $\frac{2}{3}a^2 - \frac{3}{2}ax - \frac{1}{4}x^2$ by $\frac{2}{3}a^2 + \frac{3}{2}ax + \frac{1}{4}x^2$.

51. Multiply $x^2 + (-a+b)x - ab$ by $x - c$.

52. Multiply $x^p - x^q + x^r$ by $x^p - x^q + x^r$.

53. Divide $\frac{4}{9}x^4 - \frac{1}{4}x^2 + \frac{3}{4}x - \frac{9}{16}$ by $\frac{2}{3}x^2 + \frac{1}{2}x - \frac{3}{4}$.

54. Divide $x^3 + (a-b-c)x^2 + (-ab+bc-ca)x + abc$
by $x - c$.

55. Simplify $(x+y+z)[(x+y+z)^2 - 3(xy+yz+zx)]$.

56. Simplify $(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$.

VII. SIMPLE EQUATIONS.

63. The *First Member* of an equation is the expression to the left of the sign of equality, and the *Second Member* is the expression to the right of that sign.

Thus, in the equation $2x - 3 = 3x + 5$, the first member is $2x - 3$, and the second member is $3x + 5$.

Any term of either member of an equation is called a *term* of the equation.

The *sides* of an equation are its two members.

64. An **Identical Equation,** or **Identity,** is one whose members are equal, whatever values are given to the letters involved; as $(a + b)(a - b) = a^2 - b^2$.

65. An equation is said to be *satisfied* by a set of values of certain letters involved in it when, on substituting the value of each letter wherever it occurs, the equation becomes identical.

Thus, the equation $x - y = 5$ is satisfied by the set of values $x = 8$, $y = 3$; for on substituting 8 for x, and 3 for y, the equation becomes

$$8 - 3 = 5, \text{ or } 5 = 5;$$

which is identical.

66. An **Equation of Condition** is an equation involving one or more letters, called *unknown quantities,* which is not satisfied by every set of values of these letters.

Thus, the equation $x + 2 = 5$ is not satisfied by every value of x, but only by the value $x = 3$.

An equation of condition is usually called an *equation.*

67. If an equation contains but one unknown quantity any value of the unknown quantity which satisfies the equation is called a **Root** of the equation.

Thus, 3 is a root of the equation $x + 2 = 5$.

To *solve* an equation is to find its roots.

68. A **Numerical Equation** is one in which all the known numbers are represented by Arabic numerals; as,

$$2x - 17 = x - 5.$$

69. A monomial is said to be *rational and integral* when it is either a number expressed in Arabic numerals, or a single letter with unity for its exponent, or the product of two or more such numbers or letters.

Thus, 3, a, and $2a^3bc^2$ are rational and integral.

70. If each term of an equation, involving but one unknown quantity x, is rational and integral, and no term contains a higher power of x than the first, the equation is said to be of the *first degree*.

$$\left.\begin{array}{ll}\text{Thus,} & 3x - 5 = 4 \\ \text{and} & a^2x + b^3 = c\end{array}\right\} \text{are equations of the first degree.}$$

A **Simple Equation** is an equation of the first degree.

PROPERTIES OF EQUATIONS.

71. It follows from § 9, 1 and 3, that:

1. *The same number may be added to, or subtracted from, both members of an equation, without destroying the equality.*

2. *Both members of an equation may be multiplied, or divided, by the same number, without destroying the equality.*

72. Transposition of Terms.

A term may be transposed from one member of an equation to the other by changing its sign.

Let the equation be $x + a = b$.

Subtracting a from both members (§ 71, 1), we have

$$x = b - a.$$

In this case, the term $+ a$ has been transposed from the first member to the second by changing its sign.

Again, consider the equation

$$x - a = b.$$

Adding a to both members, we have

$$x = b + a.$$

In this case, the term $- a$ has been transposed from the first member to the second by changing its sign.

73. It follows from § 72 that

If the same term occurs in both members of an equation affected with the same sign, it may be cancelled.

74. *The sign of each term of an equation may be changed without destroying the equality.*

Let the equation be $a - x = b - c$. (1)

Transposing each term (§ 72), we have

$$-b + c = -a + x.$$

That is, $x - a = c - b$;

which is the same as (1) with the sign of each term changed.

SOLUTION OF SIMPLE EQUATIONS.

75. 1. Solve the equation

$$5x - 7 = 3x + 1.$$

Transposing $3x$ to the first member, and -7 to the second, we have

$$5x - 3x = 7 + 1.$$

Uniting similar terms, $2x = 8$,

Dividing both members by 2 (§ 71, 2), we have

$$x = 4, \textit{ Ans.}$$

From the above example, we derive the following rule:

Transpose the unknown terms to the first member, and the known terms to the second.

Unite the similar terms, and divide both members by the coefficient of the unknown quantity.

2. Solve the equation

$$14 - 5x = 19 + 3x.$$

Transposing, $-5x - 3x = 19 - 14.$

Uniting terms, $-8x = 5.$

Dividing by -8, $x = -\frac{5}{8}$, *Ans.*

Note 1. The result may be *verified* by putting $x = -\frac{5}{8}$ in the given equation; thus,

$$14 - 5\left(-\frac{5}{8}\right) = 19 + 3\left(-\frac{5}{8}\right).$$

That is, $14 + \frac{25}{8} = 19 - \frac{15}{8}.$

Or, $\frac{137}{8} = \frac{137}{8}$; which is identical.

EXAMPLES.

Solve the following, in each case verifying the answer:

3. $9x = 7x + 28.$
4. $8x - 5 = -61.$
5. $6x + 11 = x + 31.$
6. $9x - 7 = 3x - 37.$
7. $4x - 3 = 8x + 33.$
8. $12 - 13x = 6 - 10x.$
9. $5x + 9 = 14 - 2x.$
10. $7x - 29 = 16x - 17.$
11. $13 - 6x = 13x - 6.$
12. $19 - 16x = 27 - 28x.$
13. $9x - 23 = 20x - 18.$
14. $30 + 17x = 27x + 22.$
15. $24x - 11 = 28 + 11x.$
16. $33x + 25 = 41 + 51x.$
17. $14x + 21 - 35 = -29x + 44x - 22.$

18. $32x - 39 = 25x - 10x - 141.$

19. $12x - 23x + 55 = 15x - 75.$

20. Solve the equation

$$(2x - 1)^2 = 2(x + 3)(2x - 3) - 3(6x - 1).$$

Expanding (Note 2), $4x^2 - 4x + 1 = 4x^2 + 6x - 18 - 18x + 3.$

Transposing,

$$4x^2 - 4x - 4x^2 - 6x + 18x = -18 + 3 - 1.$$

Uniting terms, $8x = -16.$

Dividing by 8, $x = -2$, *Ans.*

Note 2. To *expand* an algebraic expression is to perform the operations indicated.

Solve the following equations:

21. $2(5x + 1) - 4 = 3(x - 7) - 16.$

22. $10x - (3x + 2) = 9x - (5x - 4).$

23. $8x - 5(4x + 3) = -3 - 4(2x - 7).$

24. $5x - 6(3 - 4x) = x - 7(4 + x).$

25. $6x(3x - 5) + 141 = 2x(9x + 1) + 13.$

26. $19 - 5x(4x + 1) = 40 - 10x(2x - 1).$

27. $2(4x + 7) - 8(3x - 4) = 6(2x + 3) - 7(2x - 3).$

28. $(5x + 7)(3x - 8) = (5x + 4)(3x - 5).$

29. $(4x - 7)^2 = (2x - 5)(8x + 3).$

30. $(5 - 3x)(3 + 4x) - (7 + 3x)(1 - 4x) = -1.$

31. $(1 - 3x)^2 - (x + 5)^2 = 4(x + 1)(2x - 3).$

32. $6(4 - x)^2 - 5(2x + 7)(x - 2) = 5 - (2x + 3)^2.$

PROBLEMS.

76. For the solution of problems by algebraic methods, no general rule can be given, as much must depend upon the skill and ingenuity of the student.

The following suggestions will, however, be found of service:

1. Represent the unknown quantity, or one of the unknown quantities if there are several, by x.

2. Every problem contains, explicitly or implicitly, *precisely as many distinct statements as there are unknown quantities involved.*

All but one of these should be used to express the other unknown quantities in terms of x.

3. The remaining statement should then be used to form an equation.

The beginner will find it useful to write out the various statements of the problem, as shown in Exs. 1 and 2, § 77; after a little practice he will be able to dispense with these aids to the solution.

77. 1. Divide 45 into two parts such that the less part shall be one-fourth of the greater.

Here there are *two* unknown quantities, the greater part and the less.

In accordance with the first suggestion of § 76, we will represent the less part by x.

The two statements of the problem are, implicitly:

1. The sum of the greater part and the less part is 45.
2. The greater part is 4 times the less part.

In accordance with the second suggestion of § 76, we will use the *second* statement to express the greater part in terms of x.

Thus, the greater part will be represented by $4x$.

We now in accordance with the third suggestion of § 76 use the *first* statement to form an equation.

Thus, $4x + x = 45.$

Uniting terms, $5x = 45.$

Dividing by 5, $x = 9$, the less part.

Whence, $4x = 36$, the greater part.

2. A had twice as much money as B; but after giving B \$35, he had only one-third as much as B. How much had each at first?

Here there are two unknown quantities: the number of dollars A had at first, and the number B had at first.

Let x represent the number of dollars B had at first.

The first statement of the problem is:

A had twice as much money as B at first.

Then $2x$ will represent the number of dollars A had at first.

The second statement of the problem is, implicitly:

After A gives B \$35, B has 3 times as much money as A.

Now after giving B \$35, A has $2x - 35$ dollars, and B $x + 35$ dollars; we then have the equation

$$x + 35 = 3(2x - 35).$$

Expanding, $x + 35 = 6x - 105.$

Transposing, $-5x = -140.$

Dividing by -5, $x = 28$, the number of dollars B had at first;

and $2x = 56$, the number of dollars A had at first.

Note 1. It must be carefully borne in mind that x can only represent an *abstract number;* thus, in Ex. 2, we do not say, "let x represent *what* B had at first," nor "let x represent the *sum* that B had at first," but "let x represent the *number of dollars* that B had at first."

3. A is 3 times as old as B, and 8 years ago he was 7 times as old as B. Required their ages at present.

Let $x =$ the number of years in B's age.

Then, $3x =$ the number of years in A's age.

Also, $x - 8 =$ the number of years in B's age 8 years ago,

and $3x - 8 =$ the number of years in A's age 8 years ago.

But A's age 8 years ago was 7 times B's age 8 years ago.

Whence, $3x - 8 = 7(x - 8).$

Expanding, $3x - 8 = 7x - 56.$

Transposing, $-4x = -48.$

Dividing by -4, $x = 12$, the number of years in B's age.

Whence, $3x = 36$, the number of years in A's age.

Note 2. In Ex. 3, we do not say, "let x represent B's *age*," but "let x represent the *number of years* in B's age."

4. A sum of money amounting to $4.32 consists of 108 coins, all dimes and cents; how many are there of each kind?

Let $x =$ the number of dimes.

Then, $108 - x =$ the number of cents.

Also, the x dimes are worth $10x$ cents.

But the entire sum amounts to 432 *cents*.

Whence, $10x + 108 - x = 432.$

Transposing, $9x = 324.$

Whence, $x = 36$, the number of dimes;

and $108 - x = 72$, the number of cents.

PROBLEMS.

5. Divide 19 into two parts such that 7 times the less shall exceed 6 times the greater by 3.

6. What two numbers are those whose sum is 246, and whose difference is 72?

7. Divide 38 into two parts such that twice the greater shall be less by 22 than 5 times the less.

8. Divide $22 among A, B, and C, so that A may receive $2.25 more than B, and $1.75 less than C.

9. A is 5 times as old as B, and in 13 years he will be only 3 times as old as B. What are their ages?

10. B is twice as old as A, and 35 years ago he was 7 times as old as A. What are their ages?

11. A had one-third as much money as B; but after B had given him $24, he had three times as much money as B. How much had each at first?

12. A sum of money, amounting to $2.20, consists entirely of five-cent pieces and twenty-five-cent pieces, there being in all 16 coins. How many are there of each kind?

13. A is 68 years of age, and B is 11. In how many years will A be 4 times as old as B?

14. A is 25 years of age, and B is 65. How many years is it since B was 6 times as old as A?

15. A man has two kinds of money; dimes and fifty-cent pieces. If he is offered $4.10 for 17 coins, how many of each kind must he give?

16. Divide 76 into two parts such that if the greater be taken from 61, and the less from 43, the remainders shall be equal.

17. What two numbers are those whose sum is 13, and the difference of whose squares is 65?

18. Find two numbers whose difference is 6, and the difference of whose squares is 120.

19. A is 14 years younger than B; and he is as much below 60 as B is above 40. Required their ages.

20. A drover sold a certain number of oxen at $60 each, and 3 times as many cows at $35, realizing $1485 from the sale. How many of each did he sell?

21. A man has $4.35 in dollars, dimes, and cents. He has one-fourth as many dollars as dimes, and 5 times as many cents as dollars. How many has he of each kind?

22. A garrison of 4375 men contains 4 times as many cavalry as artillery, and 7½ times as many infantry as cavalry. How many are there of each kind?

23. At an election where 5760 votes were cast for three candidates, A, B, and C, B received 5 times as many votes as A, and C received twice as many votes as A and B together. How many votes did each receive?

24. Divide $115 among A, B, C, and D, so that A and B together may have $43, A and C $65, and A and D $57

25. A man divided $ 1656 among his wife, three daughters, and two sons. The wife received 4 times as much as either of the daughters, and each son one-third as much as each daughter. How much did each receive?

26. Divide $ 125 among A, B, C, and D, so that A and B together may have $ 65, B and C $ 52, and B and D $ 54.

27. A man has 4 shillings in three-penny pieces and farthings; and he has 23 more farthings than three-penny pieces. How many has he of each kind?

28. Divide 71 into two parts such that one shall be 4 times as much below 55 as the other exceeds 37.

29. A square court has the same area as a rectangular court, whose length is 9 yards greater, and width 6 yards less, than the side of the square. Find the area of the court.

30. Two men, 84 miles apart, setting out at the same time, travel towards each other at the rates of 3 and 4 miles an hour, respectively. After how many hours will they meet?

31. Find three consecutive numbers whose sum is 108.

32. In 7 years, A will be 3 times as old as B, and 8 years ago he was 6 times as old. What are their ages?

(Let x represent the number of years in B's age 8 years ago.)

33. A sum of money, amounting to $ 24.90, consists entirely of $ 2 bills, fifty-cent pieces, and dimes; there are 5 more fifty-cent pieces than $ 2 bills, and 3 times as many dimes as $ 2 bills. How many are there of each kind?

34. Find two consecutive numbers such that the difference of their squares, plus 5 times the greater number, exceeds 4 times the less number by 27.

35. Find four consecutive numbers such that the product of the first and third shall be less than the product of the second and fourth by 9.

36. A laborer agreed to serve for 32 days on condition that for every day he worked he should receive $1.75, and for every day he was absent he should forfeit $1. At the end of the time he received $28.50. How many days did he work, and how many days was he absent?

37. A merchant has grain worth 5 shillings a bushel, and other grain worth 9 shillings a bushel. In what proportion must he mix 24 bushels, so that the mixture may be worth 8 shillings a bushel?

38. A general, arranging his men in a square, finds that he has 43 men left over. But on attempting to add 1 man to each side of the square, he finds that he requires 108 men to fill up the square. Required the number of men on a side at first, and the whole number of men.

39. In a school of 535 pupils, there are 40 more pupils in the second class than in the first, and one-half as many in the first as in the third. The number in the fourth class is less by 30 than 3 times the number in the first class. How many are there in each class?

40. A man gave to a crowd of beggars 15 cents each, and found that he had 80 cents left. If he had attempted to give them 20 cents each, he would have had too little money by 10 cents. How many beggars were there?

41. A tank containing 120 gallons can be filled by two pipes, A and B, in 12 and 15 minutes, respectively. The pipe A was opened for a certain number of minutes; it was then closed, and the pipe B opened; and in this way the tank was filled in 13 minutes. How many minutes was each pipe open?

42. A grocer has tea worth 70 cents a pound, and other tea worth 40 cents a pound. In what proportion must he mix 50 pounds, so that the mixture may be worth 49 cents a pound?

VIII. IMPORTANT RULES IN MULTIPLICATION AND DIVISION.

78. Let it be required to square $a + b$.

[illegible]

$$\begin{array}{r} a + b \\ \underline{a + b} \\ a^2 + ab \\ \underline{ab + b^2} \end{array}$$

Whence, $(a + b)^2 = a^2 + 2\,ab + b^2.$

That is, *the square of the sum of two quantities is equal to the square of the first, plus twice the product of the two, plus the square of the second.*

Example. Square $3\,a + 2\,bc$.

We have, $(3\,a + 2\,bc)^2 = (3\,a)^2 + 2 \times 3\,a \times 2\,bc + (2\,bc)^2$
$= 9\,a^2 + 12\,abc + 4\,b^2c^2$, *Ans.*

79. Let it be required to square $a - b$.

$$\begin{array}{r} a - b \\ \underline{a - b} \\ a^2 - ab \\ \underline{- ab + b^2} \end{array}$$

Whence, $(a - b)^2 = a^2 - 2\,ab + b^2.$

That is, *the square of the difference of two quantities is equal to the square of the first, minus twice the product of the two, plus the square of the second.*

Example. Square $4\,x - 5$.

We have, $(4\,x - 5)^2 = (4\,x)^2 - 2 \times 4\,x \times 5 + 5^2$
$= 16\,x^2 - 40\,x + 25$, *Ans.*

80. Let it be required to multiply $a+b$ by $a-b$.

$$\begin{array}{rr} a+b & \\ a-b & \\ \hline a^2+ab & \\ -ab & -b^2 \\ \hline \end{array}$$

Whence, $(a+b)(a-b)=a^2 \qquad -b^2$.

That is, *the product of the sum and difference of two quantities is equal to the difference of their squares.*

Example. Multiply $6a^2+b$ by $6a^2-b$.

We have, $(6a^2+b)(6a^2-b)=(6a^2)^2-b^2=36a^4-b^2$, *Ans.*

81. In connection with the examples of the present chapter, a rule for raising a monomial to any power whose exponent is a positive integer will be found convenient.

Let it be required to raise $5a^2b^3c$ to the third power.

We have, $(5a^2b^3c)^3=5a^2b^3c \times 5a^2b^3c \times 5a^2b^3c=125a^6b^9c^3$.

We then have the following rule:

Raise the numerical coefficient to the required power, and multiply the exponent of each letter by the exponent of the required power.

EXAMPLES.

82. Find by inspection the values of the following:

1. $(x+4)^2$.
2. $(a-3)^2$.
3. $(6a-5b)^2$.
4. $(2xy+9)^2$.
5. $(3m+4n)(3m-4n)$.
6. $(7-2a^3)^2$.
7. $(5x^2+8)(5x^2-8)$.
8. $(a^4-6a)^2$.
9. $(8+3m^3n^2)^2$.
10. $(ab^4+2a^2b^5)^2$.
11. $(6xy-7xz)^2$.
12. $(4a^2+11bc)^2$.
13. $(9xy^3+2z^2)(9xy^3-2z^2)$.
14. $(7ab-5cd)^2$.
15. $(6x^5+11y^6)(6x^5-11y^6)$.
16. $(9a^3+5a^4)^2$.

17. $(7m^3+12n)(7m^3-12n)$. **20.** $(3a^m+4b^n)^2$.

18. $(8x^3+7yz^3)^2$. **21.** $(5x^p-8x^3)^2$.

19. $(13a^4x-6b^3y^2)^2$. **22.** $(a^{2q}+a^r)(a^{2q}-a^r)$.

23. Multiply $a+b+c$ by $a+b-c$.

$$\begin{aligned}(a+b+c)(a+b-c)&=[(a+b)+c][(a+b)-c]\\&=(a+b)^2-c^2 && (\S\ 80)\\&=a^2+2ab+b^2-c^2,\ \textit{Ans.} && (\S\ 78)\end{aligned}$$

24. Multiply $a+b-c$ by $a-b+c$.

$$\begin{aligned}(a+b-c)(a-b+c)&=[a+(b-c)][a-(b-c)]\\&=a^2-(b-c)^2 && (\S\ 80)\\&=a^2-(b^2-2bc+c^2) && (\S\ 79)\\&=a^2-b^2+2bc-c^2,\ \textit{Ans.}\end{aligned}$$

Expand the following:

25. $(a+b+c)(a-b+c)$. **28.** $(a^2+a-1)(a^2-a+1)$.

26. $(x-y+z)(x-y-z)$. **29.** $(x^2+x-2)(x^2-x-2)$.

27. $(a+b+c)(a-b-c)$. **30.** $(1+a+b)(1-a-b)$.

31. $(x^2+2x+1)(x^2-2x+1)$.

32. $(a+2b-3c)(a-2b+3c)$.

33. $(a^2+ab+b^2)(a^2-ab+b^2)$.

34. $(3x+4y+2z)(3x-4y-2z)$.

83. We find by multiplication:

$$\begin{array}{rrr}x&+5&\\x&+3&\\\hline x^2&+5x&\\&+3x&+15\\\hline x^2&+8x&+15\end{array}\qquad\qquad\begin{array}{rrr}x&-5&\\x&-3&\\\hline x^2&-5x&\\&-3x&+15\\\hline x^2&-8x&+15\end{array}$$

$$\begin{array}{rrr}x&+5&\\x&-3&\\\hline x^2&+5x&\\&-3x&-15\\\hline x^2&+2x&-15\end{array}\qquad\qquad\begin{array}{rrr}x&-5&\\x&+3&\\\hline x^2&-5x&\\&+3x&-15\\\hline x^2&-2x&-15\end{array}$$

In these results it will be observed that:

I. The coefficient of x is the *algebraic sum* of the second terms of the multiplicand and multiplier.

II. The last term is the *product* of the second terms of the multiplicand and multiplier.

By aid of the above laws, the product of any two binomials of the form $x + a$, $x + b$ may be written by inspection.

1. Required the value of $(x + 8)(x - 5)$.

The coefficient of x is $+ 8 - 5$, or 3.

The last term is $8 \times (-5)$, or -40.

Whence, $(x + 8)(x - 5) = x^2 + 3x - 40$, *Ans.*

2. Required the value of $(a - b - 3)(a - b - 4)$.

The coefficient of $a - b$ is $-3 - 4$, or -7.

The last term is $(-3) \times (-4)$, or 12.

Whence, $(a - b - 3)(a - b - 4) = (a - b)^2 - 7(a - b) + 12$, *Ans.*

EXAMPLES.

Find by inspection the values of the following:

3. $(x + 6)(x + 4)$.

4. $(x - 2)(x + 3)$.

5. $(x - 10)(x - 1)$.

6. $(x + 5)(x - 6)$.

7. $(a + 1)(a + 9)$.

8. $(a - 7)(a + 4)$.

9. $(m + 5)(m - 1)$.

10. $(x^2 - 7)(x^2 - 2)$.

11. $(n^3 + 3)(n^3 - 10)$.

12. $(ab + 2)(ab + 11)$.

13. $(xy - 12)(xy - 3)$.

14. $(a + b - 7)(a + b + 8)$.

15. $(x - 5a)(x - 11a)$.

16. $(x + y)(x - 2y)$.

17. $(a + 11b)(a - 6b)$.

18. $(a + 7x)(a + 5x)$.

19. $(x - y - 4)(x - y + 10)$.

20. $(x - 11z)(x + 9z)$.

21. $(x^2 + 3y)(x^2 + 8y)$.

22. $(x^3 - 9m^2)(x^3 - 6m^2)$.

23. $(ab + 8cd)(ab - 12cd)$.

24. $(x + y + 12)(x + y - 9)$.

84. We have by § 80,

$$\frac{a^2-b^2}{a+b}=a-b, \text{ and } \frac{a^2-b^2}{a-b}=a+b.$$

That is, *if the difference of the squares of two quantities be divided by the* **sum** *of the quantities, the quotient is the* **difference** *of the quantities.*

If the difference of the squares of two quantities be divided by the **difference** *of the quantities, the quotient is the* **sum** *of the quantities.*

1. Divide $16a^2b^4-9$ by $4ab^2+3$.

We have, $16a^2b^4=(4ab^2)^2$. (§ 81)

Whence, $\frac{16a^2b^4-9}{4ab^2+3}=4ab^2-3$, *Ans.*

EXAMPLES.

Write by inspection the values of the following:

2. $\frac{x^2-1}{x+1}$.

3. $\frac{4-a^2}{2-a}$.

4. $\frac{16m^2-49}{4m-7}$.

5. $\frac{25a^4-36}{5a^2+6}$.

6. $\frac{9x^2-16y^2}{3x+4y}$.

7. $\frac{25a^2-b^4}{5a-b^2}$.

8. $\frac{1-64m^2n^2}{1+8mn}$.

9. $\frac{4a^2b^6-c^4}{2ab^3-c^2}$.

10. $\frac{49m^2-100n^6}{7m-10n^3}$.

11. $\frac{81y^2-196x^4}{9y+14x^2}$.

12. $\frac{121b^2c^2-64a^2}{11bc+8a}$.

13. $\frac{144x^2y^4-169z^6}{12xy^2-13z^3}$.

14. $\frac{225a^{10}-64b^{12}c^8}{15a^5+8b^6c^4}$.

85. We find by actual division:

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2,$$

and

$$\frac{a^3-b^3}{a-b}=a^2+ab+b^2.$$

That is, *if the sum of the cubes of two quantities be divided by the sum of the quantities, the quotient is the square of the first quantity,* **minus** *the product of the two, plus the square of the second.*

If the difference of the cubes of two quantities be divided by the difference of the quantities, the quotient is the square of the first quantity, **plus** *the product of the two, plus the square of the second.*

1. Divide $1 + 8a^3$ by $1 + 2a$.

We have,
$$\frac{1+8a^3}{1+2a} = \frac{1+(2a)^3}{1+2a} \qquad (§\ 81)$$
$$= 1 - 2a + (2a)^2$$
$$= 1 - 2a + 4a^2, \textit{ Ans.}$$

2. Divide $27x^3 - 64y^3$ by $3x - 4y$.

We have,
$$\frac{27x^3 - 64y^3}{3x - 4y} = \frac{(3x)^3 - (4y)^3}{3x - 4y}$$
$$= (3x)^2 + (3x)(4y) + (4y)^2$$
$$= 9x^2 + 12xy + 16y^2, \textit{ Ans.}$$

EXAMPLES.

Find the values of the following:

3. $\dfrac{a^3 - 1}{a - 1}$.

4. $\dfrac{1 + x^3}{1 + x}$.

5. $\dfrac{m^3 + 8}{m + 2}$.

6. $\dfrac{27 - a^3}{3 - a}$.

7. $\dfrac{x^6 + y^6}{x^2 + y^2}$.

8. $\dfrac{a^6b^3 - c^9}{a^2b - c^3}$.

9. $\dfrac{1 + 64m^3}{1 + 4m}$.

10. $\dfrac{216 - x^3}{6 - x}$.

11. $\dfrac{a^3 + 125}{a + 5}$.

12. $\dfrac{1 - 343a^3b^6}{1 - 7ab^2}$.

13. $\dfrac{8x^3 - 125y^6}{2x - 5y^2}$.

14. $\dfrac{a^3b^3 + 512c^3}{ab + 8c}$.

15. $\dfrac{64m^3n^3 + 343}{4mn + 7}$.

16. $\dfrac{729a^6 - 125x^3}{9a^2 - 5x}$.

17. $\dfrac{512x^9y^3 + 27z^6}{8x^3y + 3z^2}$.

86. We find by actual division:

$$\frac{a^4-b^4}{a+b}=a^3-a^2b+ab^2-b^3,$$

$$\frac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3,$$

$$\frac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4,$$

$$\frac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4;\text{ etc.}$$

In these results we observe the following laws:

I. The number of terms is the same as the exponent of a in the dividend.

II. The exponent of a in the first term is less by 1 than its exponent in the dividend, and decreases by 1 in each succeeding term.

III. The exponent of b in the second term is 1, and increases by 1 in each succeeding term.

IV. If the divisor is $a-b$, all the terms of the quotient are positive; if the divisor is $a+b$, the terms of the quotient are alternately positive and negative.

87. The following principles are of great importance.

If n is any positive integer, it will be found that:

I. *a^n-b^n is always divisible by $a-b$.*

Thus, a^2-b^2, a^3-b^3, a^4-b^4, etc., are divisible by $a-b$.

II. *a^n-b^n is divisible by $a+b$ if n is even.*

Thus, a^2-b^2, a^4-b^4, a^6-b^6, etc., are divisible by $a+b$.

III. *a^n+b^n is divisible by $a+b$ if n is odd.*

Thus, a^3+b^3, a^5+b^5, a^7+b^7, etc., are divisible by $a+b$.

IV. *a^n+b^n is divisible by neither $a+b$ nor $a-b$ if n is even.*

Thus, a^2+b^2, a^4+b^4, a^6+b^6, etc., are divisible by neither $a+b$ nor $a-b$.

88. 1. Divide $a^7 - b^7$ by $a - b$.

Applying the laws of § 86, we have,

$$\frac{a^7 - b^7}{a - b} = a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6, \text{ Ans.}$$

2. Divide $16x^4 - 81$ by $2x + 3$.

We have, $$\frac{16x^4 - 81}{2x + 3} = \frac{(2x)^4 - 3^4}{2x + 3}$$

$$= (2x)^3 - (2x)^2 \times 3 + (2x) \times 3^2 - 3^3$$

$$= 8x^3 - 12x^2 + 18x - 27, \text{ Ans.}$$

EXAMPLES.

Find the values of the following:

3. $\frac{a^4 - 1}{a + 1}$.
4. $\frac{x^5 - 1}{x - 1}$.
5. $\frac{a^6 - b^6}{a + b}$.
6. $\frac{1 - x^6}{1 - x}$.
7. $\frac{a^8 - b^8}{a^2 - b^2}$.
8. $\frac{x^{10}y^5 + z^{15}}{x^2y + z^3}$.
9. $\frac{16 - x^4}{2 - x}$.
10. $\frac{1 - 16a^4}{1 + 2a}$.
11. $\frac{a^7 + b^7}{a + b}$.
12. $\frac{1 - m^7}{1 - m}$.
13. $\frac{32 + a^5}{2 + a}$.
14. $\frac{m^8 - n^8}{m + n}$.
15. $\frac{64a^6 - b^6}{2a - b}$.
16. $\frac{81x^4 - y^4}{3x + y}$.
17. $\frac{a^5 - 243x^5}{a - 3x}$.
18. $\frac{81a^4 - 256b^4}{3a - 4b}$.
19. $\frac{243x^5 + 32y^5}{3x + 2y}$.
20. $\frac{128m^7 - n^{14}}{2m - n^2}$.

IX. FACTORING.

89. To **Factor** an algebraic expression is to find two or more expressions which, when multiplied together, will produce the given expression.

90. A **Common Factor** of two or more expressions is an expression which will exactly divide each of them.

91. A monomial can always be factored; thus,

$$12a^3bc^2 = 2 \times 2 \times 3 \times a \times a \times a \times b \times c \times c.$$

It is not always possible to factor a polynomial; but there are certain types which can always be factored, the more important of which will be taken up in the present chapter.

92. Case I. *When the terms of the expression have a common monomial factor.*

1. Factor $14xy^4 - 35x^3y^2$.

Each term contains the monomial factor $7xy^2$.

Dividing the expression by $7xy^2$, the quotient is $2y^2 - 5x^2$.

Whence, $14xy^4 - 35x^3y^2 = 7xy^2(2y^2 - 5x^2)$, *Ans.*

EXAMPLES.

Factor the following:

2. $a^3 + 4a$.

3. $6x^4 - 14x^3$.

4. $30m^2 - 5m^3$.

5. $15a^3b^2 + 6ab^5$.

6. $56x^3y^2 - 32x^4y^3$.

7. $12a^4 - 20a^3 + 4a^2$.

8. $a^4b^2c^2 + a^3b^2c^3 + a^4bc^3$.

9. $12x^3y^5 + 24xy^3 - 42x^2y^6$.

10. $14a^5b^4 + 21a^4b^3 - 49a^3b^2$.

11. $81m^4n + 54m^3n^2 + 9m^2n^3$.

12. $48x^4y^2 - 144x^3y^3 + 108x^2y^4$.

13. $70a^4x^5 - 126a^2x^4 - 112a^5x^6$.

93. Case II. *When the expression is the sum of two binomials which have a common binomial factor.*

1. Factor $ac - bc + ad - bd$.

By § 92, $(ac - bc) + (ad - bd) = c(a - b) + d(a - b)$.

The two binomials have the common factor $a - b$.

Dividing the expression by $a - b$, the quotient is $c + d$.

Whence, $ac - bc + ad - bd = (a - b)(c + d)$, *Ans.*

If the third term of the given expression is *negative*, as in the following example, it is convenient to enclose the last two terms in a parenthesis preceded by a $-$ sign.

2. Factor $6x^3 - 15x^2 - 8x + 20$.

$$\begin{aligned}6x^3 - 15x^2 - 8x + 20 &= (6x^3 - 15x^2) - (8x - 20)\\ &= 3x^2(2x - 5) - 4(2x - 5)\\ &= (2x - 5)(3x^2 - 4), \textit{ Ans.}\end{aligned}$$

EXAMPLES.

Factor the following:

3. $ab + an + bm + mn$.
4. $ax - ay + bx - by$.
5. $ac - ad - bc + bd$.
6. $a^3 + a^2 + a + 1$.
7. $4x^3 - 5x^2 - 4x + 5$.
8. $2 + 3a - 8a^2 - 12a^3$.
9. $3x^3 + 6x^2 + x + 2$.
10. $10mx - 15nx - 2m + 3n$.
11. $a^3x + abcx - a^2by - b^2cy$.
12. $a^2bc - ac^2d + ab^2d - bcd^2$.
13. $30a^3 - 12a^2 - 55a + 22$.
14. $56 - 32x + 21x^2 - 12x^3$.
15. $a^3b^3 + a^2bcd^2 + ab^2c^2d + c^3d^3$.
16. $3ax - ay - 9bx + 3by$.
17. $4x^3 + x^2y^2 - 16xy - 4y^3$.
18. $20ac + 15bc + 4ad + 3bd$.
19. $16mx - 56my + 10nx - 35ny$.
20. $45a^3 - 20a^2b^2 - 63ab + 28b^3$.

94. If an expression can be resolved into two equal factors, it is said to be a *perfect square*, and one of the equal factors is called its *square root.*

Thus, since $9a^4b^2$ is equal to $3a^2b \times 3a^2b$, it is a perfect square, and $3a^2b$ is its square root.

Note. $9a^4b^2$ is also equal to $(-3a^2b) \times (-3a^2b)$; so that $-3a^2b$ is also its square root. In the examples of the present chapter, we shall consider the *positive* square root only.

95. The following rule for extracting the square root of a perfect monomial square is evident from § 94:

Extract the square root of the numerical coefficient, and divide the exponent of each letter by 2.

Thus, the square root of $25a^4b^6c^2$ is $5a^2b^3c$.

96. It follows from §§ 78 and 79 that a trinomial is a perfect square when its first and last terms are perfect squares and positive, and the second term twice the product of their square roots.

Thus, in the expression $4x^2 - 12xy + 9y^2$, the square root of the first term is $2x$, and of the last term $3y$; and the second term is equal to $2(2x)(3y)$.

Whence, $4x^2 - 12xy + 9y^2$ is a perfect square.

97. To find the square root of a perfect trinomial square, we simply reverse the rules of §§ 78 and 79:

Extract the square roots of the first and last terms, and connect the results by the sign of the second term.

Thus, the square root of $4x^2 - 12xy + 9y^2$ is $2x - 3y$.

98. Case III. *When the expression is a perfect trinomial square* (§ 96).

1. Factor $a^2 + 2ab^2 + b^4$.

By § 97, the square root of the expression is $a + b^2$.

Whence, $a^2 + 2ab^2 + b^4 = (a + b^2)^2 = (a + b^2)(a + b^2)$, *Ans.*

2. Factor $25x^2 - 40xy^3 + 16y^6$.

The square root of the expression is $5x - 4y^3$.

Whence, $25x^2 - 40xy^3 + 16y^6 = (5x - 4y^3)^2$

$= (5x - 4y^3)(5x - 4y^3)$, *Ans.*

Note. The expression may be written $16y^6 - 40xy^3 + 25x^2$; in which case, according to the rule, its square root is $4y^3 - 5x$.

Thus, another form of the result is

$$16y^6 - 40xy^3 + 25x^2 = (4y^3 - 5x)(4y^3 - 5x).$$

EXAMPLES.

Factor the following:

3. $m^2 + 2mn + n^2$.

4. $a^2 - 2ab + b^2$.

5. $9 + 6x + x^2$.

6. $a^2 - 8a + 16$.

7. $49x^2 + 14xy + y^2$.

8. $m^2 - 10mn + 25n^2$.

9. $4a^4 - 4a^2b^3c + b^6c^2$.

10. $m^2x^2 + 18mx + 81$.

11. $4a^2 - 20ax + 25x^2$.

12. $9a^2 + 42ab + 49b^2$.

13. $81x^2 - 72xy + 16y^2$.

14. $x^8 + 12x^4yz + 36y^2z^2$.

15. $64a^2b^2 + 16abcd + c^2d^2$.

16. $100x^8 - 60x^6 + 9x^4$.

17. $49m^4 + 112m^2n^3 + 64n^6$.

18. $121a^2b^2 + 132abc^2 + 36c^4$.

19. $144x^6y^2 - 120x^4y^4 + 25x^2y^6$.

20. $64a^2b^2 + 176ab^2c + 121b^2c^2$.

21. $49x^2y^2 - 168xyz^2 + 144z^4$.

22. $36a^4x^2 - 156a^3x^3 + 169a^2x^4$.

23. $(a+b)^2 - 4(a+b) + 4$.

24. $(x-y)^2 + 10(x-y) + 25$.

25. $16(a+x)^2 + 8(a+x) + 1$.

26. $4(a-b)^2 - 12(a-b) + 9$.

99. Case IV. *When the expression is the difference of two perfect squares.*

By § 80, $a^2 - b^2 = (a+b)(a-b)$.

Hence, to obtain the factors, we reverse the rule of § 80:

Extract the square root of the first square, and of the second square; add the results for one factor, and subtract the second result from the first for the other.

1. Factor $36a^2-49b^4$.

The square root of $36a^2$ is $6a$, and of $49b^4$ is $7b^2$.

Whence, $36a^2-49b^4=(6a+7b^2)(6a-7b^2)$, *Ans.*

EXAMPLES.

Factor the following:

2. a^2-b^4.
3. x^2-1.
4. $9-m^2$.
5. $16x^2-y^2$.
6. $4a^2-25$.
7. $1-36a^2b^2$.
8. $49m^4-16n^2$.
9. $25a^2-64b^2c^2$.
10. $100x^2y^2-9z^4$.
11. $64m^4-81n^6$.
12. $121a^2b^2-4c^2d^2$.
13. $81x^6-100y^6$.
14. $144m^4n^2-49$.
15. $36a^6-169x^8$.
16. $81x^{10}-196y^6z^2$.
17. $64a^{12}b^8-225c^{10}$.
18. $169-144x^8y^{14}$.
19. $196a^2x^6-121b^4y^8$.

20. Factor $(2x-3y)^2-(x-y)^2$.

We have,
$$\begin{aligned}&(2x-3y)^2-(x-y)^2\\&=[(2x-3y)+(x-y)][(2x-3y)-(x-y)]\\&=(2x-3y+x-y)(2x-3y-x+y)\\&=(3x-4y)(x-2y),\ \textit{Ans.}\end{aligned}$$

Factor the following:

21. $(a+b)^2-c^2$.
22. $(m-n)^2-x^2$.
23. $a^2-(b-c)^2$.
24. $x^2-(y+z)^2$.
25. $m^2-(n-p)^2$.
26. $(7x-2y)^2-y^2$.
27. $(a-b)^2-(x+y)^2$.
28. $(a+b)^2-(c-d)^2$.
29. $(a-x)^2-(b-y)^2$.
30. $(x+y)^2-(m+n)^2$.
31. $(8a-5)^2-(3a+7)^2$.
32. $(4x+1)^2-(x+6)^2$.
33. $(7a-5b)^2-(5a-2b)^2$.
34. $(9x+8y)^2-(2x-3y)^2$.

A polynomial may sometimes be expressed in the form of the difference of two perfect squares, when it may be factored by the rule of Case IV.

35. Factor $2mn + m^2 - 1 + n^2$.

Since $2mn$ is the middle term of a perfect trinomial square whose first and third terms are m^2 and n^2 (§ 96), we arrange the given expression so that the first, second, and last terms shall be grouped together, in the order $m^2 + 2mn + n^2$; thus,

$$\begin{aligned} 2mn + m^2 - 1 + n^2 &= (m^2 + 2mn + n^2) - 1 \\ &= (m+n)^2 - 1, \text{ by Case III.} \\ &= (m+n+1)(m+n-1), \textit{ Ans.} \end{aligned}$$

36. Factor $12y + x^2 - 9y^2 - 4$.

We have,
$$\begin{aligned} 12y + x^2 - 9y^2 - 4 &= x^2 - 9y^2 + 12y - 4 \\ &= x^2 - (9y^2 - 12y + 4) \\ &= x^2 - (3y-2)^2, \text{ by Case III.} \\ &= [x + (3y-2)][x - (3y-2)] \\ &= (x + 3y - 2)(x - 3y + 2), \textit{ Ans.} \end{aligned}$$

37. Factor $a^2 - c^2 + b^2 - d^2 - 2cd - 2ab$.

We have,
$$\begin{aligned} & a^2 - c^2 + b^2 - d^2 - 2cd - 2ab \\ &= a^2 - 2ab + b^2 - c^2 - 2cd - d^2 \\ &= (a^2 - 2ab + b^2) - (c^2 + 2cd + d^2) \\ &= (a-b)^2 - (c+d)^2, \text{ by Case III.} \\ &= [(a-b) + (c+d)][(a-b) - (c+d)] \\ &= (a - b + c + d)(a - b - c - d), \textit{ Ans.} \end{aligned}$$

Factor the following:

38. $a^2 - 2ab + b^2 - c^2$.

39. $m^2 + 2mn + n^2 - p^2$.

40. $a^2 - x^2 - 2xy - y^2$.

41. $x^2 - y^2 - z^2 + 2yz$.

42. $b^2 - 4 + 2ab + a^2$.

43. $2mn - n^2 + 1 - m^2$.

44. $9a^2 - 24ab + 16b^2 - 4c^2$.

45. $16x^2 - 4y^2 + 20yz - 25z^2$.

46. $4n^2 + m^2 - x^2 - 4mn$.

47. $4a^2 - 6b - 9 - b^2$.

48. $10xy - 9z^2 + y^2 + 25x^2$.

49. $a^2 - 2ab + b^2 - c^2 + 2cd - d^2$.

50. $a^2 - b^2 + x^2 - y^2 + 2ax + 2by$.

51. $x^2 + m^2 - y^2 - n^2 - 2mx - 2ny$.

52. $2xy - a^2 + x^2 - 2ab - b^2 + y^2$.

53. $4a^2 + 4ab + b^2 - 9c^2 + 12c - 4$.

54. $16y^2 - 36 - 8xy - z^2 + x^2 - 12z$.

55. $m^2 - 9n^2 + 25a^2 - b^2 - 10am + 6bn$.

100. Case V. *When the expression is a trinomial of the form $x^2 + ax + b$.*

We have by § 83,

$$(x+5)(x+3) = x^2 + 8x + 15,$$

$$(x-5)(x-3) = x^2 - 8x + 15,$$

$$(x+5)(x-3) = x^2 + 2x - 15,$$

and

$$(x-5)(x+3) = x^2 - 2x - 15.$$

In certain cases it is possible to reverse the process, and resolve a trinomial of the form $x^2 + ax + b$ into two binomial factors.

The first term of each factor will obviously be x; and to obtain the second terms, we simply reverse the rule of § 83.

Find two numbers whose algebraic sum is the coefficient of x, and whose product is the last term.

1. Factor $x^2 + 14x + 45$.

We find two numbers whose sum is 14, and product 45.

By inspection, we determine that the numbers are 9 and 5.

Whence, $x^2 + 14x + 45 = (x+9)(x+5)$, ***Ans.***

2. Factor $x^2 - 5x + 4$.

We find two numbers whose sum is -5, and product 4.

Since the sum is negative, and the product positive, the numbers must both be negative.

By inspection, we determine that the numbers are -4 and -1.

Whence, $x^2 - 5x + 4 = (x-4)(x-1)$, ***Ans.***

3. Factor $x^2 + 6x - 16$.

We find two numbers whose sum is 6, and product -16.

Since the sum is positive, and product negative, the numbers must be of opposite sign, and the positive number must have the greater absolute value.

By inspection, we determine that the numbers are $+8$ and -2.

Whence, $x^2 + 6x - 16 = (x + 8)(x - 2)$, *Ans.*

4. Factor $x^2 - x - 42$.

We find two numbers whose sum is -1, and product -42.

The numbers must be of opposite sign, and the negative number must have the greater absolute value.

By inspection, we determine that the numbers are -7 and $+6$.

Whence, $x^2 - x - 42 = (x - 7)(x + 6)$, *Ans.*

Note. In case the numbers are large, we may proceed as follows:

Required the numbers whose sum is -26, and product -192.

One number must be $+$, and the other $-$.

Taking in order, beginning with the factors $+1 \times -192$, all possible pairs of factors of -192, one of which is $+$ and the other $-$, we have:

$$\begin{aligned} &+1 \times -192, \\ &+2 \times -\ \ 96, \\ &+3 \times -\ \ 64, \\ &+4 \times -\ \ 48, \\ &+6 \times -\ \ 32. \end{aligned}$$

Since the sum of $+6$ and -32 is -26, they are the numbers required.

EXAMPLES.

Factor the following:

5. $x^2 + 6x + 8$.
6. $x^2 - 13x + 22$.
7. $x^2 + 6x - 7$.
8. $x^2 - 4x - 21$.
9. $x^2 - 11x + 24$.
10. $x^2 + 8x - 20$.
11. $x^2 - x - 6$.
12. $x^2 + 10x + 9$.
13. $a^2 - 7a - 44$.
14. $a^2 + a - 2$.
15. $m^2 + 11m + 30$.
16. $n^2 - 7n + 6$.

17. $x^2 + 3x - 40$.

18. $y^2 + 18y + 77$.

19. $a^2 - 15a + 54$.

20. $m^2 - 2m - 48$.

21. $c^2 + 15c + 36$.

22. $x^2 - 12x + 32$.

23. $x^2 - 6x - 55$.

24. $n^2 + 2n - 63$.

25. $m^2 - 18m + 72$.

26. $a^2 - 3a - 70$.

27. $x^2 + 4x - 96$.

28. $x^2 + 24x + 95$.

29. $b^2 - 10b - 24$.

30. $c^2 + 20c + 84$.

31. $z^2 - 21z + 110$.

32. $x^4 + 17x^2 - 84$.

33. $a^4 + 25a^2 + 150$.

34. $m^6 - 5m^3 - 36$.

35. $n^8 + 10n^4 - 96$.

36. $x^2y^2 - 19xy + 84$.

37. $a^2b^2 + 28ab + 160$.

38. $x^4y^2 - 27x^2y + 50$.

39. $a^4x^4 + 5a^2x^2 - 126$.

40. $m^2n^6 - 11mn^3 - 152$.

41. $(a+b)^2 + 23(a+b) + 60$.

42. $(x-y)^2 + 3(x-y) - 180$.

43. $(a-b)^2 - 22(a-b) + 112$.

44. $(x+y)^2 - 2(x+y) - 143$.

45. Factor $x^2 + 6abx - 27a^2b^2$.

We find two quantities whose sum is $6ab$, and product $-27a^2b^2$.
By inspection, we determine that the quantities are $-3ab$ and $9ab$.
Whence, $x^2 + 6abx - 27a^2b^2 = (x - 3ab)(x + 9ab)$, *Ans.*

46. Factor $1 - 3a - 88a^2$.

We find two quantities whose sum is $-3a$, and product $-88a^2$.
By inspection, we determine that the quantities are $8a$ and $-11a$.
Whence, $1 - 3a - 88a^2 = (1 + 8a)(1 - 11a)$, *Ans.*

Factor the following:

47. $a^2 + 12ab + 35b^2$.

48. $x^2 - 11ax + 28a^2$.

49. $x^2 + 4xy - 5y^2$.

50. $1 - 2a - 3a^2$.

51. $a^2 + 5am - 66m^2$.

52. $m^2 + 16mn + 48n^2$.

53. $x^2 - mx - 12m^2$.

54. $1 - 14a + 33a^2$.

55. $a^2 - 4ab - 60b^2$.

56. $1 + x - 72x^2$.

57. $x^2 - 15xy + 50y^2$.

58. $x^2 + 20ax + 99a^2$.

59. $m^2 - 16mn + 15n^2$.

60. $a^2 - ab - 20b^2$.

61. $1 + 18ab + 80a^2b$.

62. $x^2 + 7xy - 60y^2$.

63. $a^2b^2 + 16abc + 28c^2$.

64. $x^2 - 21xyz + 108y^2z^2$.

65. $1 + 11xy - 26x^2y^2$.

66. $a^6 - 6a^3bc^2 - 160b^2c^4$.

101. If an expression can be resolved into three equal factors, it is said to be a *perfect cube*, and one of the equal factors is called its *cube root*.

Thus, since $27a^6b^3$ is equal to $3a^2b \times 3a^2b \times 3a^2b$, it is a perfect cube, and $3a^2b$ is its cube root.

102. The following rule for extracting the cube root of a perfect monomial cube is evident from § 101:

Extract the cube root of the numerical coefficient, and divide the exponent of each letter by 3.

Thus, the cube root of $125a^6b^9c^3$ is $5a^2b^3c$.

103. Case VI. *When the expression is the sum or difference of two perfect cubes.*

By § 85, the sum or difference of two perfect cubes is divisible by the sum or difference, respectively, of their cube roots.

In either case, the quotient may be obtained by aid of the rules of § 85.

1. Factor $a^3 + 1$.

The cube root of a^3 is a, and of 1 is 1; hence, one factor is $a + 1$.
Dividing $a^3 + 1$ by $a + 1$, the quotient is $a^2 - a + 1$ (§ 85).
Whence, $a^3 + 1 = (a + 1)(a^2 - a + 1)$, *Ans.*

2. Factor $27x^3 - 64y^3$.

The cube root of $27x^3$ is $3x$, and of $64y^3$ is $4y$ (§ 102).
Hence, one factor is $3x - 4y$.
Dividing $27x^3 - 64y^3$ by $3x - 4y$, the quotient is $9x^2 + 12xy + 16y^2$ (§ 85).
Whence, $27x^3 - 64y^3 = (3x - 4y)(9x^2 + 12xy + 16y^2)$, *Ans.*

EXAMPLES.

Factor the following:

3. $m^3 + n^3$.

4. $a^3 - b^3$.

5. $a^3 - 1$.

6. $x^3 - y^3z^3$.

7. $a^6 + x^6$.

8. $1 + m^6$.

9. $64x^3 + 1$.

10. $1 - 125a^3$.

11. $27x^3 - 8y^3$.

12. $8a^3b^3 + 125$.

13. $216 - m^3$.

14. $125 - 64x^3y^3$.

15. $m^6 + 343n^3$.

16. $125b^3 - 216c^3$.

17. $343m^3 + 8x^3$.

18. $27a^6 + 343b^6$.

19. $512x^3 + 27y^3z^6$.

20. $64a^3b^6 - 729c^9$.

104. Case VII. *When the expression is the sum or difference of two equal odd powers of two quantities.*

By § 87, the sum or difference of two equal odd powers of two quantities is divisible by the sum or difference, respectively, of the quantities.

In either case, the quotient may be obtained by aid of the rules of § 86.

1. Factor $a^5 + b^5$.

By § 87, one factor is $a + b$.

Dividing $a^5 + b^5$ by $a + b$, the quotient is

$$a^4 - a^3b + a^2b^2 - ab^3 + b^4. \qquad (\S\ 86)$$

Hence, $a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$, *Ans.*

EXAMPLES.

Factor the following:

2. $x^5 - y^5$.

3. $a^5 + 1$.

4. $1 - m^5$.

5. $x^5y^5 + z^5$.

6. $a^7 + b^7$.

7. $1 - x^7$.

8. $m^7 + 1$.

9. $32 - a^5$.

10. $1 + 32x^5$.

11. $243m^5 - 1$.

12. $x^7 - 128$.

13. $32a^5 + 243b^5$.

105. By application of the rules already given, an expression may often be resolved into more than two factors.

If the terms of the expression have a common monomial factor, the method of Case I should always be applied first.

1. Factor $2ax^3y^2 - 8axy^4$.

We have, $2ax^3y^2 - 8axy^4 = 2axy^2(x^2 - 4y^2)$, by Case I.

Whence by Case IV,

$$2ax^3y^2 - 8axy^4 = 2axy^2(x + 2y)(x - 2y), \textit{ Ans.}$$

If the given expression is in the form of the difference of two perfect squares, it is always better to apply first the method of Case IV.

2. Factor $a^6 - b^6$.

We have, $a^6 - b^6 = (a^3 + b^3)(a^3 - b^3)$, by Case IV.

Whence by Case VI,

$$a^6 - b^6 = (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2), \textit{ Ans.}$$

3. Factor $x^8 - y^8$.

We have,
$$\begin{aligned} x^8 - y^8 &= (x^4 + y^4)(x^4 - y^4), \text{ by Case IV} \\ &= (x^4 + y^4)(x^2 + y^2)(x^2 - y^2) \\ &= (x^4 + y^4)(x^2 + y^2)(x + y)(x - y), \textit{ Ans.} \end{aligned}$$

MISCELLANEOUS AND REVIEW EXAMPLES.

106. Factor the following:

1. $35a^4b^2 + 98a^2b^3 - 49a^3b$.
2. $25a^2m^4 - 81b^2n^6$.
3. $x^2 + 11x + 18$.
4. $a^2bc + ac^2d - ab^2d - bcd^2$.
5. $6x^6 - 6x^3$.
6. $49m^2 + 56mn + 16n^2$.
7. $a^2 - 10a + 24$.
8. $x^3 + 17x^2 - 38x$.
9. $a^2 - (b + c)^2$.
10. $4a^5b^2 + 4a^2b^5$.
11. $a^2 + 15ab + 56b^2$.
12. $x^2y^2 - 23xy + 132$.
13. $108x^4 - 36x^3 + 3x^2$.
14. $64a^3b - 121a^5b^3$.
15. $x^6 - 1$.
16. $x^3 + x^2y + xy^2 + y^3$.
17. $a^2b^4 - 3ab^2 - 180$.
18. $2x^2 + 20xy - 78y^2$.

19. $30x^7-55x^6+65x^5-20x^4$.
20. $1-a^8$.
21. $16x^4-1$.
22. $64a^2b^2-80abc+25c^2$.
23. $15ac+18ad-35bc-42bd$.
24. $100x^6-49y^4z^8$.
25. $27a^3-64x^3$.
26. $32x^5+y^{10}$.
27. $8a^3b-72a^2b^2+162ab^3$.
28. $1-11mn-60m^2n^2$.
29. $(x-y)^2-(m-n)^2$.
30. $(1+n^2)^2-4n^2$.

31. $64x^5y^3z^3-56x^4y^3z^4+72x^5y^2z^4$.

32. $3a^6b^2-3ab^7$.
33. m^4-81.
34. $8x^3y^3+125$.
35. $(m+n)^2+7(m+n)-144$.
36. $a^2x^2-15abxy-54b^2y^2$.
37. $25x^2+110xy+121y^2$.
38. $4a^6-8a^5-2a^4+4a^3$.
39. $(5x-8y)^2-(4x-9y)^2$.
40. $5x^9+5x^3$.
41. $(a^2+9)^2-36a^2$.
42. $x^4-(x+2)^2$.
43. $a^2c^2-4b^2c^2-9a^2d^2+36b^2d^2$.
44. $(x^2-5x)^2-2(x^2-5x)-24$.
45. $16x^4-72x^2y^2+81y^4$.
46. a^6-2a^3+1.
47. $64-x^6$.
48. $45x^5+18x^4+60x^3+24x^2$.
49. $9(m-n)^2-12(m-n)+4$.
50. $9x^2+25y^2-16z^2+30xy$.
51. $343m^3+216n^3$.
52. $(9a^2+4)^2-144a^2$.
53. $(x^2+x-9)^2-9$.
54. $(a^2-2a)^2+2(a^2-2a)+1$.
55. $a^3b^3+a^3y^3-b^3x^3-x^3y^3$.
56. x^8-256.
57. $36a^2-4b^2-49c^2+28bc$.
58. m^4-625.
59. $(x^2+3x)^2+4(x^2+3x)+4$.
60. a^6-7a^3-8.
61. $27a^6-1000b^9c^{12}$.
62. $128-m^7$.
63. $2a^2bc-2b^3c-4b^2c^2-2bc^3$.
64. $(a^2+7a)^2+4(a^2+7a)-96$.
65. $x^{10}+2x^5+1$.
66. $(x^2-4)^2-(x+2)^2$.
67. $(a^2-b^2+c^2)^2-4a^2c^2$.

68. Resolve x^4-y^4 into two factors, one of which is $x+y$.

69. Resolve a^6-b^6 into two factors, one of which is $a-b$.

70. Resolve $x^9 + y^9$ into two factors by the method of § 104.

71. Resolve $x^9 + y^9$ into three factors by the method of § 103.

72. Resolve $1 - m^9$ into two factors by the method of § 104.

73. Resolve $a^9 - 1$ into three factors by the method of § 103.

74. Factor $3(m+n)^2 - 2(m^2 - n^2)$.

$$\begin{aligned} 3(m+n)^2 - 2(m^2-n^2) &= 3(m+n)^2 - 2(m+n)(m-n) \\ &= (m+n)[3(m+n) - 2(m-n)] \\ &= (m+n)(3m+3n-2m+2n) \\ &= (m+n)(m+5n), \textit{ Ans.} \end{aligned}$$

75. Factor $(a+b)^3 - (a-b)^3$.

By the method of § 103, we have

$$\begin{aligned} &(a+b)^3 - (a-b)^3 \\ &\quad = [(a+b)-(a-b)][(a+b)^2 + (a+b)(a-b) + (a-b)^2] \\ &\quad = (a+b-a+b)(a^2+2ab+b^2+a^2-b^2+a^2-2ab+b^2) \\ &\quad = 2b(3a^2+b^2), \textit{ Ans.} \end{aligned}$$

Factor the following:

76. $(m-x)^3 + 8x^3$.

77. $a^3 - (a-b)^3$.

78. $5(x^2-y^2) + 4(x-y)^2$.

79. $(a^3+b^3) - 2ab(a+b)$.

80. $a^2+b^2-c^2-d^2+2ab-2cd$.

81. $(x+1)^3 + (x-1)^3$.

82. $(x^3+y^3) + x(x+y)^2$.

83. $a^6 - a^4 - a^2 + 1$.

84. $a^2-b^2+x^2-y^2+2ax+2by$.

85. $(x-m)^3 - x(x^2-m^2)$.

86. $(x+y)^3 - (x-y)^3$.

87. $a^{10} - 1$.

88. $x^7 + x^4 - x^3 - 1$.

89. $(a^3-1) - (a-1)^3$.

90. $(3m-2)^3 + (2m+1)^3$.

91. $(x^2-y^2-z^2)^2 - 4y^2z^2$.

92. $a^2 + 25b^2 - 16c^2 - 9d^2 - 10ab - 24cd$.

93. $(1+a^3) + 2(1-a)(1+a)^2$.

X. HIGHEST COMMON FACTOR.

107. The **Degree** of a rational and integral monomial (§ 69) is the number of letters which are multiplied together to form its literal portion.

Thus, $2a$ is of the *first* degree; $5ab$ of the *second* degree; $3a^2b^3$, being the same as $3aabbb$, is of the *fifth* degree; etc.

The *degree* of a rational and integral monomial is equal to the sum of the exponents of the letters involved in it.

Thus, a^4bc^3 is of the *eighth* degree.

108. A polynomial is said to be *rational and integral* when each term is rational and integral; as $2a^2b - 3c + d^2$.

The *degree* of a rational and integral polynomial is the degree of its term of highest degree.

Thus, $2a^2b - 3c + d^2$ is of the *third* degree.

109. A **Prime Factor** of an expression is a factor which cannot be divided without a remainder by any expression except itself and unity.

Thus, the prime factors of $6a^2(x^2 - 1)$ are $2, 3, a, a, x + 1$, and $x - 1$.

110. The **Highest Common Factor** (H. C. F.) of two or more expressions is the product of all their common prime factors.

It is evident from this definition that the highest common factor of two or more expressions is the expression of *highest degree* (§ 108) which will divide each of them without a remainder.

111. Two expressions are said to be *prime to each other* when unity is their highest common factor.

112. Required the H. C. F. of $a^4b^2c^3$, $a^2b^3c^5$, and a^3bc^4.

Resolving each expression into its prime factors, we have

$$a^4b^2c^3 = aaaabbccc,$$
$$a^2b^3c^5 = aabbbccccc,$$
and
$$a^3bc^4 = aaabcccc.$$

Here the common prime factors are a, a, b, c, c, and c.

Whence, the H. C. F. $= aabccc = a^2bc^3$.

It will be observed, in the above result, that *the exponent of each letter is the lowest exponent with which it occurs in any of the given expressions.*

113. In determining the highest common factor of algebraic expressions, we may distinguish two cases.

114. Case I. *When the expressions are monomials, or polynomials which can be readily factored by inspection.*

1. Find the H. C. F. of $28\,a^2b^3$, $42\,ab^5c$, and $98\,a^3b^4d^2$.

We have,
$$28\,a^2b^3 = 2^2 \times 7 \times a^2b^3,$$
$$42\,ab^5c = 2 \times 3 \times 7 \times ab^5c,$$
and
$$98\,a^3b^4d^2 = 2 \times 7^2 \times a^3b^4d^2.$$

By the rule of § 112, the H. C. F. $= 2 \times 7 \times ab^3 = 14\,ab^3$, *Ans.*

EXAMPLES.

Find the highest common factor of:

2. $2\,a^4b$, $5\,a^3b^2$.
3. $20\,x^2y$, $15\,xy^3$.
4. $45\,a^2b^5$, $120\,a^3c^4$.
5. $182\,x^3yz^2$, $84\,x^2y^3z$.
6. $16\,m^2n^4$, $56\,m^4n^2$, $88\,m^3n^3$.
7. $36\,a^5bc^3$, $72\,a^3b^2c$, $180\,ab^3c^6$.
8. $126\,a^3x^5$, $21\,a^2x^7y^2$, $147\,a^4x^6z$.
9. $140\,m^5n^2x^2$, $175\,m^8n^3y$, 105
10. $117\,a^7b^2c^6$, $104\,a^4b^3c^5$, 156

11. Find the H. C. F. of

$$5x^4y - 45x^2y \text{ and } 10x^3y^2 + 40x^2y^2 - 210xy^2.$$

We have, $5x^4y - 45x^2y = 5x^2y(x^2 - 9)$

$= 5x^2y(x + 3)(x - 3)$, (§ 99)

and $10x^3y^2 + 40x^2y^2 - 210xy^2 = 10xy^2(x^2 + 4x - 21)$

$= 2 \times 5 \times xy^2(x + 7)(x - 3)$. (§ 100)

By the rule of § 112, the H. C. F. is $5xy(x - 3)$, *Ans.*

12. Find the H. C. F. of

$$4a^2 - 4a + 1,\ 8a^3 - 1, \text{ and } 2am - m - 2an + n.$$

We have, $4a^2 - 4a + 1 = (2a - 1)^2$, (§ 98)

$8a^3 - 1 = (2a - 1)(4a^2 + 2a + 1)$, (§ 103)

and $2am - m - 2an + n = (2a - 1)(m - n)$. (§ 93)

By the rule of § 112, the H. C. F. is $2a - 1$, *Ans.*

Find the highest common factor of:

13. $6a^3b^2 - 15a^2b^3,\ 12a^4b + 21a^3b^2$.

14. $68(m + n)^2(m - n)^4,\ 85(m + n)^3(m - n)$.

15. $x^2 - 9y^2,\ x^2 - 6xy + 9y^2$.

16. $3a^3 - 21a^2 - a + 7,\ a^2 + 6a - 91$.

17. $2a^3x + 4a^2x^2 + 2ax^3,\ 3a^4x + 3ax^4$.

18. $m^3 - 27,\ m^2 - 11m + 24$.

19. $ac + ad - bc - bd,\ a^2 - 6ab + 5b^2$.

20. $x + 4x^2 + 4x^3,\ 4 + 44x + 72x^2$.

21. $80n^7 - 5n^3,\ 20n^4 + 5n^2$.

22. $a^2 + b^2 - c^2 + 2ab,\ a^2 - b^2 - c^2 + 2bc$.

23. $x^2 + 2x - 24,\ x^2 - 14x + 40,\ x^2 - 8x + 16$.

24. $9a^2 - 12a + 4,\ 9a^2 - 4,\ 18a^3 - 12a^2$.

25. $x^2 - 6x - 27,\ x^2 + 6x + 9,\ x^3 + 27$.

.3)
5)

26. $a^3 + 13a^2 + 40a,\ a^4 - a^3 - 30a^2,\ a^5 + 2a^4 - 15a^3.$

27. $m^3 - 4m,\ m^3 + 9m^2 - 22m,\ 2m^4 - 4m^3 - 3m^2 + 6m.$

28. $x^3 - 8y^3,\ x^2 - 4y^2,\ x^2 - 9xy + 14y^2.$

29. $3a^3 - a^2b + 3ab - b^2,\ 27a^3 - b^3,\ 9a^2 - 6ab + b^2.$

30. $27x^3 + 125,\ 9x^2 - 25,\ 9x^2 + 30x + 25.$

31. $x^3y - x^2y^2 - 20xy^3,\ 2x^3y^2 + 22x^2y^3 + 56xy^4,\ 3x^4y - 48x^2y^3.$

32. $16m^4 - n^4,\ 16m^4 - 8m^2n^2 + n^4,\ 2mx + 2my - nx - ny.$

33. $a^5 - x^5,\ a^3 - a^2x - ax^2 + x^3,\ 3a^3 - 3a^2x + 5ax^2 - 5x^3.$

115. Case II. *When the expressions are polynomials which cannot be readily factored by inspection.*

The rule in Arithmetic for the H. C. F. of two numbers is:

Divide the greater number by the less.

If there be a remainder, divide the divisor by it; and continue thus to make the remainder the divisor, and the preceding divisor the dividend, until there is no remainder.

The last divisor is the H. C. F. *required.*

Thus, let it be required to find the H. C. F. of 169 and 546.

$$\begin{array}{l}
169)546(3 \\
\quad\ \ \underline{507} \\
\quad\ \ \ 39)169(4 \\
\qquad\quad\ \underline{156} \\
\qquad\quad\ \ 13)39(3 \\
\qquad\qquad\ \ \underline{39}
\end{array}$$

Then, 13 is the H. C. F. required.

116. We will now prove that a rule similar to that of § 115 holds for the H. C. F. of two algebraic expressions.

Let A and B be two polynomials, the degree of A (§ 108) being not lower than that of B.

Suppose that B is contained in A p times, with a remainder C; that C is contained in B q times, with a remainder D; and that D is contained in C r times, with no remainder.

To prove that D is the H. C. F. of A and B.

The operation of division is shown as follows:

$$\begin{array}{l}
B)A(p \\
\;\;\underline{pB} \\
\qquad C)B(q \\
\qquad\;\;\underline{qC} \\
\qquad\quad D)C(r \\
\qquad\qquad \underline{rD} \\
\qquad\qquad\;\; 0
\end{array}$$

We will first prove that D is a common factor of A and B.

Since the minuend is equal to the subtrahend plus the remainder (§ 35), we have

$$A = pB + C, \tag{1}$$

$$B = qC + D, \tag{2}$$

and

$$C = rD.$$

Substituting the value of C in (2), we obtain

$$B = qrD + D = D(qr + 1). \tag{3}$$

Substituting the values of B and C in (1), we have

$$A = pD(qr + 1) + rD = D(pqr + p + r). \tag{4}$$

From (3) and (4), D is a common factor of A and B.

We will next prove that every common factor of A and B is a factor of D.

Let F be any common factor of A and B; and let

$$A = mF \text{ and } B = nF.$$

From the operation of division, we have

$$C = A - pB, \tag{5}$$

and

$$D = B - qC. \tag{6}$$

Substituting the values of A and B in (5), we have

$$C = mF - pnF.$$

Substituting the values of B and C in (6), we have

$$D = nF - q(mF - pnF) = F(n - qm + pqn).$$

Whence, F is a factor of D.

Then, since every common factor of A and B is a factor of D, and since D is itself a common factor of A and B, it follows that D is the *highest* common factor of A and B.

117. Hence, to find the H. C. F. of two polynomials, A and B, of which the degree of A is not lower than that of B,

Divide A by B.

If there be a remainder, divide the divisor by it; and continue thus to make the remainder the divisor, and the preceding divisor the dividend, until there is no remainder.

The last divisor is the H. C. F. *required.*

Note 1. Each division should be continued until the remainder is of a lower degree than the divisor.

Note 2. It is of the greatest importance to arrange the given polynomials in the same order of powers of some common letter (§ 33), and also to arrange each remainder in the same order.

1. Find the H. C. F. of

$$6x^2 - 13x - 5 \text{ and } 18x^3 - 51x^2 + 13x + 5.$$

$$\begin{array}{l}
6x^2 - 13x - 5)18x^3 - 51x^2 + 13x + 5(3x - 2 \\
\qquad\qquad\qquad \underline{18x^3 - 39x^2 - 15x} \\
\qquad\qquad\qquad\qquad - 12x^2 + 28x \\
\qquad\qquad\qquad\qquad \underline{- 12x^2 + 26x + 10} \\
\qquad\qquad\qquad\qquad\qquad\qquad 2x - 5)6x^2 - 13x - 5(3x + 1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{6x^2 - 15x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2x \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{2x - 5}
\end{array}$$

Whence, $2x - 5$ is the H. C. F. required.

Note 3. If the terms of one of the given expressions have a common factor which is not a common factor of the terms of the other, it may be removed; for it can evidently form no part of the highest common factor. In like manner, we may divide any remainder by a factor which is not a factor of the preceding divisor.

2. Find the H. C. F. of

$$6x^3-25x^2+14x \text{ and } 6ax^2+11ax-10a.$$

In accordance with Note 3, we remove the factor x from the first expression, and the factor a from the second.

$$\begin{array}{r}
6x^2-25x+14)6x^2+11x-10(1\\
\underline{6x^2-25x+14}\\
36x-24
\end{array}$$

We divide this remainder by 12 (Note 3).

$$\begin{array}{l}
3x-2)6x^2-25x+14(2x-7\\
\underline{6x^2-\ \ 4x}\\
-21x\\
\underline{-21x+14}
\end{array}$$

Whence, $3x-2$ is the H. C. F. required.

Note 4. If the given expressions have a common factor which can be seen by inspection, remove it, and find the H. C. F. of the resulting expressions. The result, multiplied by the common factor, will be the H. C. F. of the given expressions.

3. Find the H. C. F. of

$$2a^3-3a^2b-2ab^2 \text{ and } 2a^3+7a^2b+3ab^2.$$

In accordance with Note 4, we remove the common factor a, and find the H. C. F. of $2a^2-3ab-2b^2$ and $2a^2+7ab+3b^2$.

$$\begin{array}{r}
2a^2-3ab-2b^2)2a^2+\ \ 7ab+3b^2(1\\
\underline{2a^2-\ \ 3ab-2b^2}\\
5b)\underline{10ab+5b^2}\\
2a+b
\end{array}$$

$$\begin{array}{l}
2a+b)2a^2-3ab-2b^2(a-2b\\
\underline{2a^2+\ \ ab}\\
-4ab\\
\underline{-4ab-2b^2}
\end{array}$$

Multiplying $2a+b$ by a, the required H. C. F. is $a(2a+b)$.

Note 5. If the first term of the dividend, or of any remainder, is not divisible by the first term of the divisor, it may be made so by multiplying the dividend or remainder by any term which is not a factor of the divisor.

Note 6. If the first term of any remainder is negative, the sign of each term of the remainder may be changed.

4. Find the H. C. F. of

$$2x^3 - 3x^2 + 2x - 8 \text{ and } 3x^3 - 7x^2 + 4x - 4.$$

$$
\begin{array}{rl}
 & 3x^3 - 7x^2 + 4x - 4 \\
 & \phantom{3x^3 - 7x^2 + 4x - {}}2 \\
2x^3 - 3x^2 + 2x - 8) & \overline{6x^3 - 14x^2 + 8x - 8}(3 \\
 & \underline{6x^3 - 9x^2 + 6x - 24} \\
 & -5x^2 + 2x + 16 \\
 & \\
 & 2x^3 - 3x^2 + 2x - 8 \\
 & \phantom{2x^3 - 3x^2 + 2x - {}}5 \\
5x^2 - 2x - 16) & \overline{10x^3 - 15x^2 + 10x - 40}(2x \\
 & \underline{10x^3 - 4x^2 - 32x} \\
 & -11x^2 + 42x - 40 \\
 & \underline{\phantom{-11x^2 + 42x - {}}5} \\
 & -55x^2 + 210x - 200(-11 \\
 & \underline{-55x^2 + 22x + 176} \\
 & 188)\underline{188x - 376} \\
 & x - 2
\end{array}
$$

$$
\begin{array}{rl}
x - 2) & 5x^2 - 2x - 16(5x + 8 \\
 & \underline{5x^2 - 10x} \\
 & 8x \\
 & \underline{8x - 16}
\end{array}
$$

Whence, $x - 2$ is the H. C. F. required.

In the above example, we multiply $3x^3 - 7x^2 + 4x - 4$ by 2 in order to make its first term divisible by $2x^3$.

We change the sign of each term of the first remainder (Note 6), and multiply $2x^3 - 3x^2 + 2x - 8$ by 5 to make its first term divisible by $5x^2$.

We multiply the remainder $-11x^2 + 42x - 40$ by 5 to make its first term divisible by $5x^2$.

EXAMPLES.

Find the H. C. F. of:

5. $2x^2-5x+3,\ 2x^2-7x+5.$
6. $2a^2+7a+6,\ 6a^2+11a+3.$
7. $4x^2+13x+10,\ 6x^2+5x-14.$
8. $x^2+5x-24,\ x^3+4x^2-26x+15.$
9. $3m^2+m-2,\ 4m^3+2m^2-m+1.$
10. $18a^2+9ab-5b^2,\ 24a^2-29ab+7b^2.$
11. $12a^3-5a^2x-11ax^2+6x^3,\ 15a^3+11a^2x-8ax^2-4x^3.$
12. $4x^3-12x^2+5x,\ 2x^4+x^3-7x^2-20x.$
13. $3x^3+13x^2y+12xy^2,\ 9x^3y-22xy^3-8y^4.$
14. $4a^4-11a^2+5a+12,\ 6a^6-11a^5+13a^3-4a^2.$
15. $2m^4+5m^3n-2m^2n^2+3mn^3,\ 6m^3n-7m^2n^2+5mn^3-2n^4.$
16. $3x^2-4x-4,\ 3x^4-7x^3+6x^2-9x+2.$
17. $3a^4+5a^3+12a^2+8,\ 6a^4+10a^3+19a^2-10a-4.$
18. $2m^3-3m^2x-8mx^2-3x^3,$
    $3m^4-7m^3x-5m^2x^2-mx^3-6x^4.$
19. $2a^4-a^3-4a^2+3a,\ 4a^4-6a^3+a^2+4a-3.$
20. $m^5+8m^2,\ m^5-2m^4-15m^3-14m^2.$
21. $4a^4-22a^3b+6a^2b^2+20ab^3,\ 9a^3b-42a^2b^2-18ab^3+15b^4.$
22. $4x^2+9x-9,\ 2x^4+11x^3+14x^2-5x-6.$
23. $3a^4-6a^3+4a^2+4a-4,\ 3a^5+3a^4-11a^3-2a^2+6a.$
24. $a^3+2a^2-2a+24,\ a^4+2a^3-11a^2-6a+24.$
25. $2x^4-3x^3y+3x^2y^2-3xy^3+y^4,$
    $2x^4+x^3y-3x^2y^2+5xy^3-2y^4.$
26. $2x^4+x^3-9x^2+x+1,\ 2x^4-9x^3+12x^2-3x-2.$
27. $2x^3-7x^2+7x-2,\ x^5-3x^4+5x^2-4x+4.$
28. $a^5x-a^4x^2-a^3x^3-a^2x^4-2ax^5,$
    $a^5x+3a^4x^2-a^3x^3-4a^2x^4-ax^5.$

118. The H. C. F. of three expressions may be found as follows:

Let A, B, and C be the expressions.

Let G be the H. C. F. of A and B; then, every common factor of G and C is a common factor of A, B, and C.

But since every common factor of two expressions exactly divides their highest common factor (§ 116), every common factor of A, B, and C is also a common factor of G and C.

Whence, the highest common factor of G and C is the highest common factor of A, B, and C.

Hence, *to find the* H. C. F. *of three expressions, find the* H. C. F. *of two of them, and then of this result and the third expression.*

We proceed in a similar manner to find the H. C. F. of any number of expressions.

1. Find the H. C. F. of

$$x^3 - 7x + 6,\ x^3 + 3x^2 - 16x + 12, \text{ and } x^3 - 5x^2 + 7x - 3.$$

The H. C. F. of $x^3 - 7x + 6$ and $x^3 + 3x^2 - 16x + 12$ is $x^2 - 3x + 2$.

The H. C. F. of $x^2 - 3x + 2$ and $x^3 - 5x^2 + 7x - 3$ is $x - 1$, *Ans.*

EXAMPLES.

Find the H. C. F. of:

2. $2x^2 - 17x + 36,\ 4x^2 - 12x - 27,\ 6x^2 - 31x + 18.$

3. $8a^2 + 22a + 5,\ 12a^2 - 13a - 4,\ 20a^2 + 29a + 6.$

4. $15m^2 - 4m - 32,\ 18m^2 + 3m - 28,\ 21m^2 + 25m - 4.$

5. $5a^2 + 23ab - 10b^2,\ 5a^3 + 33a^2b + 46ab^2 - 24b^3,$
$5a^3 + 38a^2b + 59ab^2 - 30b^3.$

6. $x^3 + x^2 - 14x - 24,\ x^3 - 3x^2 - 6x + 8,\ x^3 + 4x^2 + x - 6.$

7. $a^3 - a^2 - 5a - 3,\ a^3 + 2a^2 - a - 2,\ a^3 - 2a^2 - 2a + 1.$

8. $2m^3 + 9m^2 - 6m - 5,\ 3m^3 + 10m^2 - 23m + 10,$
$6m^3 - 7m^2 - m + 2.$

9. $2x^3 - x^2y - 27xy^2 + 36y^3,\ 2x^3 - 5x^2y - 37xy^2 + 60y^3,$
$2x^3 - 19x^2y + 54xy^2 - 45y^3.$

XI. LOWEST COMMON MULTIPLE.

119. A **Common Multiple** of two or more expressions is an expression which can be divided by each of them without a remainder.

120. The **Lowest Common Multiple** (L. C. M.) of two or more expressions is the product of all their different prime factors (§ 109), each taken the greatest number of times that it occurs as a factor in any one of the expressions.

121. Required the L. C. M. of a^2b^3c, ab^5d^2, and $b^2c^3d^4$.

Here, the different prime factors are a, b, c, and d; a occurs twice as a factor in a^2b^3c; b five times as a factor in ab^5d^2; c three times as a factor in $b^2c^3d^4$; and d four times as a factor in $b^2c^3d^4$.

Whence, the required L. C. M. is $a^2b^5c^3d^4$ (§ 120).

It will be observed, in the above result, that *the exponent of each letter is the highest exponent with which it occurs in any one of the given expressions.*

122. It is evident from the definition of § 120 that the lowest common multiple of two or more expressions is the expression of *lowest degree* (§ 108) which can be divided by each of them without a remainder.

123. If two expressions are prime to each other (§ 111), their product is their lowest common multiple.

124. In determining the lowest common multiple of algebraic expressions, we may distinguish two cases.

125. CASE I. *When the expressions are monomials, or polynomials which can be readily factored by inspection.*

1. Find the L. C. M. of $28a^4b^2$, $54bc^3$, and $63c^2d$.

We have, $28a^4b^2 = 2^2 \times 7 \times a^4b^2$,

$54bc^3 = 2 \times 3^3 \times bc^3$,

and $63c^2d = 3^2 \times 7 \times c^2d$.

By the rule of § 121, the L. C. M. $= 2^2 \times 3^3 \times 7 \times a^4b^2c^3d$

$= 756a^4b^2c^3d$, *Ans.*

EXAMPLES.

Find the lowest common multiple of:

2. $5ab^3$, $7a^2b^2$.

3. $12xy^3$, $54yz^2$.

4. $24m^3$, $45n^2$.

5. $72a^3b$, $96b^2c^4$.

6. $55xy$, $70yz$, $77zx$.

7. $50a^4b^3$, $60a^5b^2$, $75a^3b^4$.

8. $15x^4y^3$, $21y^5z$, $33x^2z^3$.

9. $20ab^3$, $27b^2c^3$, $90c^4d^2$.

10. $36m^5nx$, $40mn^2y^4$, $48n^3x^2y$.

11. $56a^2bc^3$, $84a^3b^6d^5$, $126a^7c^2d^3$.

12. Find the L. C. M. of $x^2 + x - 6$, $x^2 - 4x + 4$, and $x^3 - 9x$.

We have $x^2 + x - 6 = (x + 3)(x - 2)$, (§ 100)

$x^2 - 4x + 4 = (x - 2)^2$, (§ 98)

and $x^3 - 9x = x(x + 3)(x - 3)$. (§ 99)

By the rule of § 121, the L. C. M. $= x(x - 2)^2(x + 3)(x - 3)$, *Ans.*

Find the lowest common multiple of:

13. $a^2 - b^2$, $a^2 + 2ab + b^2$.

14. $m^2 + mn$, $mn - n^2$.

15. $x^2 - 9$, $x^2 + 10x + 21$.

16. $x^4 - 18x^3 + 81x^2$, $x^3 - 13x^2 + 36x$.

17. $a^2 - 3ab + 2b^2$, $ac + ad - bc - bd$.

18. $a^2 + 2ax + x^2$, $a^3 + x^3$.

19. $1 - 8x^3$, $1 + 9x - 22x^2$.

20. $m^3 + 13m^2n + 40mn^2$, $m^2n - mn^2 - 30n^3$.

21. $4x^2 - 25$, $2x^3 - 5x^2 - 4x + 10$.

22. $x^3 + 3ax^2 - 18a^2x$, $ax^2 + 15a^2x + 54a^3$.

23. $4a^2 - 2ab$, $4ab + 2b^2$, $4a^2 - b^2$.

24. $6x^2 + 10xy$, $9xy - 15y^2$, $36x^3y - 100xy^3$.

25. $4m^2 - 8m + 4$, $6m^2 + 12m + 6$, $m^2 - 1$.

26. $a^2 - 12a + 35$, $a^2 + 2a - 63$, $a^2 - 3a - 108$.

27. $x^4 - 4ax^3 + 4a^2x^2$, $x^2 + 4ax + 4a^2$, $ax^3 - 4a^3x$.

28. $3x^2 - 6x - 72$, $4x^2 + 8x - 192$, $2x^2 - 24x + 72$.

29. $x^3y - xy^3$, $x^3 - y^3$, $x^2 - 2xy + y^2$.

30. $x^2 + y^2 - z^2 - 2xy$, $x^2 - y^2 - z^2 - 2yz$.

31. $16m^2 - 9n^2$, $8ab^2m - 6ab^2n$, $16m^2 - 24mn + 9n^2$.

32. $a^3 - a$, $a^3 - 9a^2 - 10a$, $a^4 - a^3 + a^2 - a$.

33. $x^2 + 4xy + 4y^2$, $x^2 + xy - 2y^2$, $x^3 + 8y^3$.

34. $2a^3 - 2a^2 - 4a$, $3a^4 - 6a^3 - 9a^2$, $4a^5 + 20a^4 + 16a^3$.

35. $27x^3 - 8$, $9x^2 - 4$, $9x^2 - 12x + 4$.

36. $4x^2 - 4m^2$, $6x + 6m$, $8x^2 + 8m^2$, $9x - 9m$.

37. $x^4 - y^4$, $x^4 + 2x^2y^2 + y^4$, $x^4 - 2x^2y^2 + y^4$.

38. $a^3 + b^3$, $a^3 - b^3$, $(a^2 + b^2)^2 - a^2b^2$.

39. $a^2 - 11ax + 18x^2$, $a^2 - 5ax - 14x^2$, $a^4 - 8a^2x^2 + 16x^4$.

40. $m^2 - n^2$, $m^3 - m^2n - mn^2 + n^3$, $m^3 + m^2n - mn^2 - n^3$.

41. $a^2 + b^2 - c^2 + 2ab$, $a^2 - b^2 - c^2 - 2bc$, $a^2 - b^2 + c^2 - 2ac$.

126. CASE II. *When the expressions are polynomials which cannot be readily factored by inspection.*

Let A and B be any two expressions.

Let F be their H. C. F., and M their L. C. M.; and suppose that $A = aF$, and $B = bF$.

Then, $$A \times B = abF^2. \qquad (1)$$

Since F is the H. C. F. of A and B, a and b have no common factors; whence, the L. C. M. of aF and bF is abF.

That is, $$M = abF.$$

Multiplying each of these equals by F, we have

$$F \times M = abF^2. \qquad (2)$$

From (1) and (2), $$A \times B = F \times M. \qquad (\S\ 9, 4)$$

That is, *the product of two expressions is equal to the product of their* H. C. F. *and* L. C. M.

Therefore, to find the L. C. M. of two expressions,

Divide their product by their highest common factor; or,

Divide one of the expressions by their highest common factor, and multiply the quotient by the other expression.

1. Find the L. C. M. of

$$6x^2 - 17x + 12 \text{ and } 12x^2 - 4x - 21.$$

$$\begin{array}{l}
6x^2 - 17x + 12)\,12x^2 - 4x - 21\,(2 \\
\qquad\qquad\qquad\;\; \underline{12x^2 - 34x + 24} \\
\qquad\qquad\qquad\qquad 15)\,30x - 45 \\
\qquad\qquad\qquad\qquad\quad 2x - 3)\,6x^2 - 17x + 12\,(3x - 4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\;\; \underline{6x^2 - 9x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad - 8x \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{- 8x + 12}
\end{array}$$

Then the H. C. F. of the expressions is $2x - 3$.

Dividing $6x^2 - 17x + 12$ by $2x - 3$, the quotient is $3x - 4$.

Whence, the L. C. M. $= (3x - 4)(12x^2 - 4x - 21)$, *Ans.*

EXAMPLES.

Find the L. C. M. of:

2. $2x^2 - 3x - 35$, $2x^2 - 19x + 45$.

3. $3a^2 - 13a + 4$, $3a^2 + 14a - 5$.

4. $6a^2 + 25ab + 24b^2$, $12a^2 + 16ab - 3b^2$.

5. $6x^3 + 11x^2y - 2xy^2$, $8x^2y + 21xy^2 + 10y^3$.

6. $12m^2 - 21m - 45,\ 4m^3 - 11m^2 - 6m + 9.$
7. $2a^3 - 5a^2 - 18a - 9,\ 3a^3 - 14a^2 - a + 6.$
8. $2a^3x + a^2x^2 + 2ax^3 + 3x^4,\ 2a^3x + 5a^2x^2 + 2ax^3 - x^4.$
9. $2a^2 - 5ab + 3b^2,\ a^4 + a^3b - 5a^2b^2 + 2ab^3 + b^4.$
10. $6x^3 - 7x^2 + 5x - 2,\ 4x^4 - 5x^2 + 4x - 3.$
11. $2a^3 - 5a^2 + a + 2,\ 4a^3 - 9a - 4.$
12. $3m^4 - 7m^3n + 4mn^3,\ 6m^3n - 4m^2n^2 - 14mn^3 - 4n^4.$
13. $a^5 + 2a^4 - 5a^3 + 12a^2,\ 3a^6 + 11a^5 - 6a^4 - 7a^3 + 4a^2.$
14. $3x^4 - 2x^3 - 12x^2 - x + 6,\ 3x^4 + 7x^3 + 6x^2 - 2x - 4.$

127. The L. C. M. of three expressions may be found as follows:

Let A, B, and C be the expressions.

Let M be the L. C. M. of A and B; then, every common multiple of M and C is a common multiple of A, B, and C.

But since every common multiple of two expressions is exactly divisible by their lowest common multiple, every common multiple of A, B, and C is also a common multiple of M and C.

Whence, the lowest common multiple of M and C is the lowest common multiple of A, B, and C.

Hence, *to find the* L. C. M. *of three expressions, find the* L. C. M. *of two of them, and then of this result and the third expression.*

We proceed in a similar manner to find the L. C. M. of any number of expressions.

EXAMPLES.

Find the L. C. M. of:

1. $2x^2 + x - 15,\ 2x^2 + 7x + 3,\ 2x^2 + 9x + 9.$
2. $3a^2 + a - 2,\ 6a^2 + 11a + 5,\ 9a^2 + 5a - 4.$
3. $2m^2 - 5m + 2,\ 3m^2 - 10m + 8,\ 4m^2 + 10m - 6.$
4. $2x^3 - 5x^2 - 3x,\ 4x^4 - 11x^3 - 3x^2,\ 6x^5 - x^4 - 2x^3.$
5. $a^3 - 2a^2 - 5a + 6,\ a^3 - 3a^2 - a + 3,\ a^3 + 4a^2 + a - 6.$

XII. FRACTIONS.

128. The quotient of a divided by b is written $\frac{a}{b}$ (§ 3).

The expression $\frac{a}{b}$ is called a **Fraction**; the dividend a is called the *numerator*, and the divisor b the *denominator*.

The numerator and denominator are called the *terms* of the fraction.

129. Let
$$\frac{a}{b} = x. \qquad (1)$$

Then since the dividend is the product of the divisor and quotient (§ 54), we have
$$a = bx.$$

Multiplying each of these equals by c (§ 9, 1),
$$ac = bcx.$$

Regarding ac as the dividend, bc as the divisor, and x as the quotient, this may be written
$$\frac{ac}{bc} = x. \qquad (2)$$

From (1) and (2),
$$\frac{ac}{bc} = \frac{a}{b}. \qquad (§\ 9,\ 4)$$

That is, *if the terms of a fraction be both multiplied, or both divided, by the same expression, the value of the fraction is not altered.*

130. By the Law of Signs in Division (§ 55),
$$\frac{+a}{+b} = \frac{-a}{-b} = -\frac{+a}{-b} = -\frac{-a}{+b}.$$

[illegible]

That is, *if the signs of both ~~terms~~ of a fraction be changed, the sign before the fraction is not changed; but if the sign of either one be changed, the sign before the fraction is changed.*

If either ~~term~~ member is a polynomial, care must be taken, on changing its sign, to change the sign of *each of its ~~terms~~.*

Thus, the fraction $\frac{a-b}{c-d}$, by changing the signs of both numerator and denominator, can be written $\frac{b-a}{d-c}$ (§ 41).

131. It follows from §§ 49 and 130 that

If either ~~term~~ members of a fraction is the indicated product of two or more expressions, the signs of any **even** *number of ~~them~~ factors may be changed without changing the sign before the fraction; but if the signs of any* **odd** *number of them be changed, the sign before the fraction is changed.*

Thus, the fraction $\frac{a-b}{(c-d)(e-f)}$ may be written

$$\frac{a-b}{(d-c)(f-e)},\ \frac{b-a}{(d-c)(e-f)},\ -\frac{b-a}{(d-c)(f-e)},\ \text{etc.}$$

REDUCTION OF FRACTIONS.

132. To Reduce a Fraction to its Lowest Terms.

A fraction is said to be in its *lowest terms* when its numerator and denominator are prime to each other (§ 111).

133. Case I. *When the numerator and denominator can be readily factored by inspection.*

By § 129, dividing both terms of a fraction by the same expression, or cancelling common factors in the numerator and denominator, does not alter the value of the fraction.

We then have the following rule:

Resolve both numerator and denominator into their factors, and cancel all that are common to both.

1. Reduce $\frac{24a^3b^2c}{40a^2b^2d}$ to its lowest terms.

We have, $$\frac{24a^3b^2c}{40a^2b^2d}=\frac{2^3\times 3\times a^3b^2c}{2^3\times 5\times a^2b^2d}.$$

Cancelling the common factor $2^3 \times a^2b^2$, we obtain

$$\frac{24\,a^3b^2c}{40\,a^2b^2d} = \frac{3\,ac}{5\,d},\ \textit{Ans.}$$

2. Reduce $\dfrac{x^3 - 27}{x^2 - 2x - 3}$ to its lowest terms.

We have, $$\frac{x^3 - 27}{x^2 - 2x - 3} = \frac{(x-3)(x^2+3x+9)}{(x-3)(x+1)} \qquad (\S\S\ 100,\ 103)$$

$$= \frac{x^2+3x+9}{x+1},\ \textit{Ans.}$$

Note. If all the factors of the numerator be cancelled, unity remains to form a numerator; thus, $\dfrac{x^2y^3}{x^5y^4} = \dfrac{1}{x^3y}$.

If all the factors of the denominator be cancelled, the division is exact.

EXAMPLES.

Reduce each of the following to its lowest terms:

3. $\dfrac{a^5b^2c^3}{ab^4c^3}$.

4. $\dfrac{7\,m^4n^6p}{2\,m^2n^5p^4}$.

5. $\dfrac{54\,x^4y^3}{45\,y^2z^5}$.

6. $\dfrac{32\,ab^4}{72\,a^6b^5}$.

7. $\dfrac{56\,a^4m^2n^6}{84\,a^4m^6n^3}$.

8. $\dfrac{120\,x^5y^3z^2}{15\,x^4z^2}$.

9. $\dfrac{18\,ab^3}{108\,a^2b^6c^4}$.

10. $\dfrac{60\,m^5n^3x^5}{96\,m^6n^9x}$.

11. $\dfrac{126\,a^6b^2c^9}{98\,ab^{10}c^2}$.

12. $\dfrac{3\,a^3b - 6\,a^2b^2}{4\,a^2b^2 - 8\,ab^3}$.

13. $\dfrac{6\,x^7y + 8\,x^5y^4}{15\,x^4y^3 + 20\,x^2y^6}$.

14. $\dfrac{a^2 + 7\,a + 10}{a^2 + 4\,a - 5}$.

15. $\dfrac{x^3 - 8\,x^2 + 12\,x}{x^2 - 12\,x + 36}$.

16. $\dfrac{25\,a^2 + 20\,ab + 4\,b^2}{25\,a^2 - 4\,b^2}$.

17. $\dfrac{m^3 - m^2 - 56\,m}{m^4 + m^3 - 42\,m^2}$.

18. $\dfrac{x^3 + y^3}{2\,x^3y - 2\,x^2y^2 + 2\,xy^3}$.

19. $\dfrac{64\,a^3 + 112\,a^2x + 49\,ax^2}{64\,a^2x - 49\,x^3}$.

20. $\dfrac{x^2 - 14\,mx + 45\,m^2}{x^2 - 2\,mx - 15\,m^2}$.

21. $\dfrac{a^3 - 8}{a^3 - 2\,a^2 + a - 2}$.

22. $\dfrac{4m^3-10m^2-6m+15}{6m^3+8m^2-9m-12}$.

23. $\dfrac{x^2-y^2+z^2+2xz}{x^2-y^2-z^2+2yz}$.

24. $\dfrac{27a^3+64b^3}{9a^2+24ab+16b^2}$.

25. $\dfrac{(a^2-9)(a^2+5a+6)}{(a^2+6a+9)(a^2-a-6)}$.

26. $\dfrac{(a+b)^2-(c+d)^2}{(a-d)^2-(b-c)^2}$.

27. $\dfrac{12x^3+8x^2-3x-2}{18x^3-9x^2-8x+4}$.

28. Reduce $\dfrac{ax-bx-ay+by}{b^2-a^2}$ to its lowest terms.

We have, $\dfrac{ax-bx-ay+by}{b^2-a^2}=\dfrac{(a-b)(x-y)}{(b+a)(b-a)}$. (§§ 93, 99)

Changing the signs of the factors of the numerator (§ 131), we have

$$\frac{ax-bx-ay+by}{b^2-a^2}=\frac{(b-a)(y-x)}{(b+a)(b-a)}=\frac{y-x}{b+a},\ \textit{Ans.}$$

Reduce each of the following to its lowest terms:

29. $\dfrac{9-m^2}{m^2-7m+12}$.

30. $\dfrac{14x^2-4x^3}{4x^2-28x+49}$.

31. $\dfrac{x^2-7xy+6y^2}{y^2-x^2}$.

32. $\dfrac{2ac-2bc-ad+bd}{d^2-4c^2}$.

33. $\dfrac{1-11a+18a^2}{8a^3-1}$.

34. $\dfrac{a^2-(b+c)^2}{(b-a)^2-c^2}$.

134. Case II. *When the numerator and denominator cannot be readily factored by inspection.*

Since the H. C. F. of two expressions is the product of all their common prime factors (§ 110), we have the following rule:

Divide both numerator and denominator by their highest common factor.

1. Reduce $\dfrac{2a^2-5a+3}{6a^2-a-12}$ to its lowest terms.

By the rule of § 117, we find the H. C. F. of $2a^2-5a+3$ and $6a^2-a-12$ to be $2a-3$.

Dividing $2a^2-5a+3$ by $2a-3$, the quotient is $a-1$.

Dividing $6a^2-a-12$ by $2a-3$, the quotient is $3a+4$.

Whence, $$\frac{2a^2-5a+3}{6a^2-a-12}=\frac{a-1}{3a+4},\ \textit{Ans.}$$

EXAMPLES.

Reduce each of the following to its lowest terms:

2. $\dfrac{x^2-3x-18}{5x^2-23x-42}.$

3. $\dfrac{2a^2+a-10}{4a^2+8a-5}.$

4. $\dfrac{2x^2-xy-15y^2}{2x^2-15xy+27y^2}.$

5. $\dfrac{6m^2-13m+6}{9m^2+6m-8}.$

6. $\dfrac{x^2+3x-10}{x^3+2x^2-14x+5}.$

7. $\dfrac{3a^2+14ab+8b^2}{4a^2+15ab-4b^2}.$

8. $\dfrac{3x^3-17x^2+4x+4}{3x^3-14x^2-11x-2}.$

9. $\dfrac{2a^3+9a^2-2a-3}{6a^3+23a^2-22a+3}.$

10. $\dfrac{m^3+m^2+m+6}{m^3+6m^2+6m-4}.$

11. $\dfrac{a^3+2a^2x-2ax^2-x^3}{a^3-3a^2x-2ax^2+4x^3}.$

135. To Reduce a Fraction to an Integral or Mixed Expression.

An **Integral Expression** is an expression which has no fractional part; as $2xy$, or $a+b$.

An integral expression may be considered as a fraction whose denominator is 1; thus, $a+b$ is the same as $\frac{a+b}{1}$.

A **Mixed Expression** is an expression which has both integral and fractional parts; as $a+\frac{b}{c}$, or $x+\frac{y+z}{y-z}$.

136. We have by § 30,

$$a\times\left(\frac{b}{a}+\frac{c}{a}\right)=a\times\frac{b}{a}+a\times\frac{c}{a}=b+c. \qquad (\S\ 9,\ 3)$$

Regarding $b + c$ as the dividend, a as the divisor, and $\frac{b}{a} + \frac{c}{a}$ as the quotient (§ 54), this may be written

$$\frac{b+c}{a} = \frac{b}{a} + \frac{c}{a}.$$

137. A fraction may be reduced to an integral or mixed expression by the operation of division, if the degree (§ 108) of the numerator is equal to, or greater than, that of the denominator.

1. Reduce $\frac{6x^2 + 15x - 2}{3x}$ to a mixed expression.

By § 136, $\frac{6x^2 + 15x - 2}{3x} = \frac{6x^2}{3x} + \frac{15x}{3x} - \frac{2}{3x} = 2x + 5 - \frac{2}{3x}$, *Ans.*

2. Reduce $\frac{12x^3 - 8x^2 + 4x - 5}{4x^2 + 3}$ to a mixed expression.

$$\begin{array}{r@{\,}l}
4x^2 + 3) & 12x^3 - 8x^2 + 4x - 5(3x - 2 \\
 & \underline{12x^3 \qquad\quad + 9x} \\
 & \qquad\; - 8x^2 - 5x \\
 & \qquad\; \underline{- 8x^2 \qquad - 6} \\
 & \qquad\qquad\quad - 5x + 1
\end{array}$$

A remainder of lower degree than the divisor may be written over the divisor in the form of a fraction, and the result added to the quotient.

Thus, $\frac{12x^3 - 8x^2 + 4x - 5}{4x^2 + 3} = 3x - 2 + \frac{-5x + 1}{4x^2 + 3}.$

If the first term of the numerator is negative, it is usual to *change the sign of each term of the numerator*, at the same time changing the sign before the fraction (§ 130).

Thus, $\frac{12x^3 - 8x^2 + 4x - 5}{4x^2 + 3} = 3x - 2 - \frac{5x - 1}{4x^2 + 3}$, *Ans.*

EXAMPLES.

Reduce each of the following to a mixed expression:

3. $\frac{12x^3 - 16x + 7}{4x}.$ 4. $\frac{15a^3 + 6a^2 - 3a - 8}{3a}.$

5. $\frac{8x^3+1}{2x+3}$.
6. $\frac{x^3+y^3}{x-y}$.
7. $\frac{a^3-2b^3}{a+b}$.

8. $\frac{15a^3+11a^2-15a-6}{3a+4}$.

9. $\frac{4m^2}{2m-5n}$.

10. $\frac{2x^3-3x^2-5}{x^2-x-1}$.

11. $\frac{12a^2-5a-5}{4a-1}$.

12. $\frac{12m^4+19m^3-7m}{4m^2+1}$.

13. $\frac{x^4+y^4}{x+y}$.

14. $\frac{18a^3-3a^2+38}{3a^2-4a+5}$.

15. $\frac{a^5+b^5}{a-b}$.

16. $\frac{8x^4+16x^3-10x^2-28x+11}{2x^2+x-3}$.

138. To Reduce a Mixed Expression to a Fraction.

The process being the reverse of that of § 137, we have the following rule:

Multiply the integral part by the denominator.

Add the numerator to the product when the sign before the fraction is +, and subtract it when the sign is −; and write the result over the denominator.

1. Reduce $\frac{x+5}{2x-3}+x-2$ to a fractional form.

We have,
$$\frac{x+5}{2x-3}+x-2=\frac{x+5+(x-2)(2x-3)}{2x-3}$$
$$=\frac{x+5+2x^2-7x+6}{2x-3}$$
$$=\frac{2x^2-6x+11}{2x-3},\ \textit{Ans.}$$

If the numerator is a polynomial, it is convenient to enclose it in a parenthesis when the sign before the fraction is −.

2. Reduce $a-b-\dfrac{a^2-ab-b^2}{a+b}$ to a fractional form.

We have, $$a-b-\frac{a^2-ab-b^2}{a+b}=\frac{(a+b)(a-b)-(a^2-ab-b^2)}{a+b}$$
$$=\frac{a^2-b^2-a^2+ab+b^2}{a+b}$$
$$=\frac{ab}{a+b},\ \textit{Ans.}$$

EXAMPLES.

Reduce each of the following to a fractional form:

3. $a-4+\dfrac{a+2}{3a}$.

4. $\dfrac{x+y}{x-y}+1$.

5. $5a+1-\dfrac{6}{2a-3}$.

6. $3x-2-\dfrac{11x^2+7}{5x}$.

7. $1-\dfrac{3a-b}{3a+b}$.

8. $m^2-mn+n^2-\dfrac{2n^3}{m+n}$.

9. $\dfrac{2a+5x}{2a-5x}-1$.

10. $3x+4+\dfrac{9x^2+16}{3x-4}$.

11. $x+2y-\dfrac{x^2+4y^2}{x+2y}$.

12. $4m^2-9+\dfrac{6m(2m-3)}{2m+3}$.

13. $2a^2+3a-\dfrac{4a(a-2)}{2a-1}$.

14. $\dfrac{a^2-7b^2}{4a-3b}+a-5b$.

15. $x+y-\dfrac{x^3+y^3}{x^2+xy+y^2}$.

16. $a^3+a^2b+ab^2+b^3+\dfrac{2b^4}{a-b}$.

17. $\dfrac{(x^2-1)^2}{x^2+x+1}-(x^2-x+1)$.

18. $m+3n-\dfrac{m^3-27n^3}{m^2-3mn+9n^2}$.

139. To Reduce Fractions to their Lowest Common Denominator.

To reduce fractions to their **Lowest Common Denominator** (L. C. D.) is to express them as equivalent fractions, having for their common denominator the lowest common multiple of the given denominators.

Let it be required to reduce $\frac{4\,cd}{3\,a^2b}$, $\frac{3\,mx}{2\,ab^2}$, and $\frac{5\,ny}{4\,a^3b}$ to their lowest common denominator.

The L. C. M. of $3\,a^2b$, $2\,ab^2$, and $4\,a^3b$ is $12\,a^3b^2$ (§ 125).

By § 129, if both terms of a fraction be multiplied by the same expression, the value of the fraction is not altered.

Multiplying both terms of $\frac{4\,cd}{3\,a^2b}$ by $4\,ab$, both terms of $\frac{3\,mx}{2\,ab^2}$ by $6\,a^2$, and both terms of $\frac{5\,ny}{4\,a^3b}$ by $3\,b$, we have

$$\frac{16\,abcd}{12\,a^3b^2},\ \frac{18\,a^2mx}{12\,a^3b^2},\ \text{and}\ \frac{15\,bny}{12\,a^3b^2}.$$

It will be seen that the terms of each fraction are multiplied by an expression which is obtained by dividing the L. C. D. by its own denominator; whence the following rule:

Find the lowest common multiple of the given denominators. Divide this by each denominator separately, multiply the corresponding numerators by the quotients, and write the results over the common denominator.

Before applying the rule, each fraction should be reduced to its lowest terms.

140. 1. Reduce $\frac{4\,a}{a^2-4}$ and $\frac{3\,a}{a^2-5\,a+6}$ to their lowest common denominator.

We have, $a^2-4=(a+2)(a-2)$, and $a^2-5\,a+6=(a-2)(a-3)$.

Then the L. C. D. is $(a+2)(a-2)(a-3)$. (§ 125)

Dividing the L. C. D. by $(a+2)(a-2)$, the quotient is $a-3$; and dividing it by $(a-2)(a-3)$, the quotient is $a+2$.

Multiplying $4\,a$ by $a-3$, the product is $4\,a(a-3)$; and multiplying $3\,a$ by $a+2$, the product is $3\,a(a+2)$.

Then the required fractions are

$$\frac{4\,a(a-3)}{(a+2)(a-2)(a-3)}\ \text{and}\ \frac{3\,a(a+2)}{(a+2)(a-2)(a-3)},\ \textit{Ans.}$$

EXAMPLES.

Reduce the following to their lowest common denominator.

2. $\frac{5xy}{6}, \frac{3xz}{14}, \frac{4yz}{21}.$

3. $\frac{1}{2m^3n}, \frac{2}{3mn^3}, \frac{6}{5m^2n^2}.$

4. $\frac{2a-5c}{9a^2b}, \frac{4a+3b}{12ac^3}.$

5. $\frac{7az^2}{8xy^2}, \frac{9by}{10x^2z}, \frac{8cx^3}{15yz^2}.$

6. $\frac{3x}{6x^2+2x}, \frac{5x}{9x^2-1}.$

7. $\frac{ax}{x+y}, \frac{by^2}{(x+y)^2}, \frac{cx^2y}{(x+y)^3}.$

8. $\frac{2a}{a^2-b^2}, \frac{4b^3}{a^3+b^3}.$

9. $\frac{3}{a+1}, \frac{6}{a-1}, \frac{9}{a^2+1}.$

10. $\frac{2}{3x^3-12x^2}, \frac{x}{x^2-6x+8}, \frac{x^3}{x^3-8}.$

11. $\frac{x+y}{ax-bx-ay+by}, \frac{a-b}{x^2-2xy+y^2}.$

12. $\frac{a+5}{a^2-a-6}, \frac{a+3}{a^2+7a+10}, \frac{a-2}{a^2+2a-15}.$

ADDITION AND SUBTRACTION OF FRACTIONS.

141. We have by § 136,

$$\frac{b}{a}+\frac{c}{a}=\frac{b+c}{a}.$$

In like manner, $\frac{b}{a}-\frac{c}{a}=\frac{b-c}{a}.$

Whence the following rule:

To add or subtract fractions, reduce them, if necessary, to equivalent fractions having the lowest common denominator.

Add or subtract the numerator of each resulting fraction, according as the sign before the fraction is + or −, and write the result over the lowest common denominator.

The final result should be reduced to its lowest terms.

142. 1. Simplify $\frac{4a+3}{4a^2b}+\frac{1-6b^2}{6ab^3}$.

The L. C. D. is $12a^2b^3$.

Multiplying the terms of the first fraction by $3b^2$, and the terms of the second by $2a$, we have

$$\frac{4a+3}{4a^2b}+\frac{1-6b^2}{6ab^3}=\frac{12ab^2+9b^2}{12a^2b^3}+\frac{2a-12ab^2}{12a^2b^3}$$

$$=\frac{12ab^2+9b^2+2a-12ab^2}{12a^2b^3}=\frac{9b^2+2a}{12a^2b^3},\ \textit{Ans.}$$

If a fraction whose numerator is a polynomial is preceded by a − sign, it is convenient to enclose the numerator in a parenthesis preceded by a − sign, as shown in Ex. 2.

If this is not done, care must be taken to *change the sign of each term of the numerator* before combining it with the other numerators.

2. Simplify $\frac{5x-4y}{6}-\frac{7x-2y}{14}$.

The L. C. D. is 42.

Whence, $\frac{5x-4y}{6}-\frac{7x-2y}{14}=\frac{35x-28y}{42}-\frac{21x-6y}{42}$

$$=\frac{35x-28y-(21x-6y)}{42}$$

$$=\frac{35x-28y-21x+6y}{42}$$

$$=\frac{14x-22y}{42}=\frac{7x-11y}{21},\ \textit{Ans.}$$

EXAMPLES.

Simplify the following:

3. $\frac{5a-6}{8}+\frac{3a+7}{12}$.

4. $\frac{4}{3xy^2}-\frac{6}{5x^2y}$.

5. $\frac{3x+4}{12}-\frac{2x+5}{16}$.

6. $\frac{a-4x}{6ax^2}-\frac{7x-6a}{9a^2x}$.

7. $\dfrac{x-3m}{24m}+\dfrac{4x+m}{32x}$.

8. $\dfrac{2a-b}{ab}+\dfrac{2b-c}{bc}+\dfrac{2c-a}{ca}$.

9. $\dfrac{2a-9}{7}+\dfrac{3a-5}{14}-\dfrac{4a+7}{28}$.

10. $\dfrac{x+1}{2x}-\dfrac{3x-4}{5x^2}+\dfrac{5x+7}{8x^3}$.

11. $\dfrac{5a+1}{6a}-\dfrac{2b+3}{8b}-\dfrac{7c-4}{12c}$.

12. $\dfrac{3x-y}{5x}+\dfrac{4x-5y}{10y}-\dfrac{6x^2+2y^2}{15xy}$.

13. $\dfrac{6x+1}{3}-\dfrac{5x-2}{6}+\dfrac{8x-3}{9}-\dfrac{7x+4}{12}$.

14. $\dfrac{3a+4}{3}+\dfrac{4a-3}{4}-\dfrac{5a+2}{5}-\dfrac{6a-1}{6}$.

15. $\dfrac{2a-3b}{9}-\dfrac{3a+b}{18}-\dfrac{4a-5b}{27}+\dfrac{5a+7b}{36}$.

16. Simplify $\dfrac{1}{x^2+x}-\dfrac{1}{x^2-x}$.

We have, $x^2+x=x(x+1)$, and $x^2-x=x(x-1)$.

Then the L. C. D. is $x(x+1)(x-1)$, or $x(x^2-1)$.

Multiplying the terms of the first fraction by $x-1$, and the terms of the second by $x+1$, we have

$$\frac{1}{x^2+x}-\frac{1}{x^2-x}=\frac{x-1}{x(x^2-1)}-\frac{x+1}{x(x^2-1)}$$

$$=\frac{x-1-(x+1)}{x(x^2-1)}=\frac{x-1-x-1}{x(x^2-1)}=\frac{-2}{x(x^2-1)},\ \textit{Ans.}$$

By changing the sign of the numerator, at the same time changing the sign before the fraction (§ 130), we may write the answer $-\dfrac{2}{x(x^2-1)}$.

Or, by changing the sign of the numerator, and of the factor x^2-1 of the denominator (§ 131), we may write it $\dfrac{2}{x(1-x^2)}$.

17. Simplify $\frac{1}{a^2-3a+2}-\frac{2}{a^2-4a+3}+\frac{1}{a^2-5a+6}$.

We have, $a^2-3a+2=(a-1)(a-2)$, $a^2-4a+3=(a-1)(a-3)$, and $a^2-5a+6=(a-2)(a-3)$.

Then the L. C. D. is $(a-1)(a-2)(a-3)$.

Whence, $\frac{1}{a^2-3a+2}-\frac{2}{a^2-4a+3}+\frac{1}{a^2-5a+6}$

$$=\frac{a-3}{(a-1)(a-2)(a-3)}-\frac{2(a-2)}{(a-1)(a-2)(a-3)}+\frac{a-1}{(a-1)(a-2)(a-3)}$$

$$=\frac{a-3-2(a-2)+a-1}{(a-1)(a-2)(a-3)}=\frac{a-3-2a+4+a-1}{(a-1)(a-2)(a-3)}=0, \textit{ Ans.}$$

Simplify the following:

18. $\frac{2}{3a+5}+\frac{1}{4a-7}$.

19. $\frac{m}{m-1}-\frac{1}{m+1}$.

20. $\frac{3}{2x+1}-\frac{4}{5x-6}$.

21. $\frac{a}{a+b}+\frac{b}{a-b}$.

22. $\frac{3a}{a+4}-\frac{2a^2-6a-3}{a^2-3a-28}$.

23. $\frac{m+n}{m-n}+\frac{m-n}{m+n}$.

24. $\frac{1-x}{1+x}-\frac{1+x}{1-x}$.

25. $\frac{4a^2+1}{4a^2-1}-\frac{2a-1}{2a+1}$.

26. $\frac{2x-y}{x}-\frac{y(y-3x)}{x^2-xy}$.

27. $\frac{a+b}{4a^2-9b^2}-\frac{a-b}{(2a+3b)^2}$.

28. $\frac{1}{x^2+4x-12}-\frac{1}{x^2-3x-54}$.

29. $\frac{x}{x^2-6ax+9a^2}-\frac{x}{x^2+4ax-21a^2}$.

30. $\frac{a^2+b^2}{a^2+ab}-\frac{a}{a+b}-\frac{b}{a}$.

31. $\frac{x}{1+x}+\frac{3x}{1-x}-\frac{6x^2}{1-x^2}$.

32. $\frac{a}{a-b}-\frac{b}{a+b}-\frac{2b^2}{a^2-b^2}$.

33. $\frac{x}{x-y}-\frac{y}{x+y}-1$.

34. $\dfrac{1}{a(a+x)}-\dfrac{1}{a(a-x)}+\dfrac{2x}{a^2-x^2}.$

35. $\dfrac{1}{x+2}-\dfrac{2x}{(x+2)^2}-\dfrac{3x^2+4}{(x+2)^3}.$

36. $\dfrac{2}{a-3}-\dfrac{1}{a+6}-\dfrac{1}{a}.$

37. $\dfrac{x+2}{x-2}-\dfrac{x-2}{x+2}-\dfrac{16}{x^2-4}.$

38. $\dfrac{x+y}{x-y}-\dfrac{x^3+y^3}{x^3-y^3}.$

39. $\dfrac{1}{2a+b}-\dfrac{(2a-b)^2}{8a^3+b^3}.$

40. $\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}-\dfrac{4ax}{a^2-x^2}.$

41. $1-\dfrac{2(x+y)}{x-y}+\dfrac{(x+y)^2}{(x-y)^2}.$

42. $\dfrac{1}{m-n}-\dfrac{2m}{(m-n)^2}+\dfrac{m^2+n^2}{(m-n)^3}.$

43. $\dfrac{x+1}{x+2}-\dfrac{x-3}{x-4}-\dfrac{x-5}{x^2-2x-8}.$

44. $\dfrac{1}{(a-b)(b-c)}+\dfrac{1}{(b-c)(c-a)}+\dfrac{1}{(c-a)(a-b)}.$

45. $\dfrac{x-2}{x-3}-\dfrac{x-3}{x-2}+\dfrac{1}{x^2-5x+6}.$

46. $\dfrac{1}{a+b}+\dfrac{1}{a-b}-\dfrac{2a}{a^2+b^2}.$

47. $\dfrac{1}{a-x}-\dfrac{3x}{a^2-x^2}+\dfrac{ax}{a^3-x^3}.$

48. $\dfrac{1}{a+1}-\dfrac{a}{a^2-a+1}+\dfrac{a^2-4}{a^3+1}.$ [illegible]

49. $\dfrac{x+z}{(x-y)(y-z)}-\dfrac{y+z}{(x-y)(x-z)}-\dfrac{x+y}{(x-z)(y-z)}.$

50. $\dfrac{x+2}{x^2+4x+3}-\dfrac{2(x-1)}{x^2+x-6}+\dfrac{x-3}{x^2-x-2}.$

In certain examples, the principles of §§ 130 and 131 enable us to change the form of a fraction so that the given denominators shall be arranged in the same order of powers.

51. Simplify $\frac{3}{a-b}+\frac{2b+a}{b^2-a^2}$.

Changing the signs of the terms in the second denominator, at the same time changing the sign before the fraction (§ 130), we have

$$\frac{3}{a-b}-\frac{2b+a}{a^2-b^2}.$$

The L. C. D. is now a^2-b^2.

Whence, $\frac{3}{a-b}-\frac{2b+a}{a^2-b^2}=\frac{3(a+b)-(2b+a)}{a^2-b^2}$

$$=\frac{3a+3b-2b-a}{a^2-b^2}=\frac{2a+b}{a^2-b^2}, \text{ Ans.}$$

52. Simplify $\frac{1}{(x-y)(x-z)}-\frac{1}{(y-x)(y-z)}-\frac{1}{(z-x)(z-y)}$.

By § 131, we change the sign of the factor $y-x$ in the second denominator, at the same time changing the sign before the fraction; and we change the signs of both factors of the third denominator.

The expression then becomes

$$\frac{1}{(x-y)(x-z)}+\frac{1}{(x-y)(y-z)}-\frac{1}{(x-z)(y-z)}.$$

The L. C. D. is now $(x-y)(x-z)(y-z)$; whence the result

$$=\frac{(y-z)+(x-z)-(x-y)}{(x-y)(x-z)(y-z)}=\frac{y-z+x-z-x+y}{(x-y)(x-z)(y-z)}$$

$$=\frac{2y-2z}{(x-y)(x-z)(y-z)}=\frac{2(y-z)}{(x-y)(x-z)(y-z)}=\frac{2}{(x-y)(x-z)}, \text{ Ans.}$$

Simplify the following:

53. $\frac{y}{x^2-xy}-\frac{x}{y^2-xy}$.

54. $\frac{x+5}{3x-6}-\frac{2x-7}{8-4x}$.

55. $\frac{5a}{a^2-9}+\frac{4}{3-a}$.

56. $\frac{1}{4m-m^2}+\frac{1}{m^2-16}$.

57. $\frac{a}{ab-b^2}+\frac{1}{b-a}-\frac{1}{b}$.

58. $\frac{a}{a+1}+\frac{a}{1-a}-\frac{2}{a^2-1}$.

59. $\frac{x}{2+x}-\frac{x}{2-x}-\frac{x^2}{x^2-4}$.

60. $\frac{x}{\cancel{x+y}}-\frac{y}{x-y}-\frac{2y^2}{y^2-x^2}$.

61. $\frac{1}{a} - \frac{1}{2a-3} + \frac{2a^2-9}{9a-4a^3}$. 62. $\frac{m}{m+2} - \frac{m}{m-2} - \frac{2m^2}{4-m^2}$.

63. $\frac{1}{(a-b)(a+c)} + \frac{1}{(b-a)(b+c)}$.

64. $\frac{x^2}{1-x^3} + \frac{2x}{1+x+x^2} - \frac{1}{x-1}$.

65. $\frac{1}{(x-y)(y-z)} - \frac{1}{(y-x)(x-z)} + \frac{1}{(z-x)(z-y)}$.

66. $\frac{a}{(a-b)(a-c)} + \frac{b}{(b-c)(b-a)} + \frac{c}{(c-a)(c-b)}$.

MULTIPLICATION OF FRACTIONS.

143. Required the product of $\frac{a}{b}$ and $\frac{c}{d}$.

Let $$\frac{a}{b} \times \frac{c}{d} = x. \qquad (1)$$

Multiplying each of these equals by $b \times d$ (§ 9, 1), we have

$$\frac{a}{b} \times \frac{c}{d} \times b \times d = x \times b \times d.$$

Or, since the factors of a product may be written in any order,

$$\left(\frac{a}{b} \times b\right) \times \left(\frac{c}{d} \times d\right) = x \times b \times d.$$

Whence, $$(a) \times (c) = x \times b \times d. \qquad (§\ 9,\ 3)$$

Dividing each of these equals by $b \times d$ (§ 9, 1), we have

$$\frac{a \times c}{b \times d} = x. \qquad (2)$$

From (1) and (2), $$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}. \qquad (§\ 9,\ 4)$$

We then have the following rule for the multiplication of fractions:

Multiply the numerators together for the numerator of the product, and the denominators for its denominator.

Common factors in the numerators and denominators should be cancelled before performing the multiplication.

Integral or mixed expressions should be expressed in a fractional form (§§ 135, 138), before applying the rule.

144. 1. Multiply $\frac{10a^3y}{9bx^2}$ by $\frac{3b^4x^3}{4a^3y^3}$.

We have, $\frac{10a^3y}{9bx^2}\times\frac{3b^4x^3}{4a^3y^3}=\frac{\cancel{2}\times 5\times\cancel{3}\times\cancel{a^3}b^{\cancel{4}}x^{\cancel{3}}\cancel{y}}{3^{\cancel{2}}\times 2^{\cancel{2}}\times\cancel{a^3}\cancel{b}\cancel{x^2}y^{\cancel{3}}}=\frac{5b^3x}{6y^2}$, *Ans.*

In this case, the factors cancelled are 2, 3, a^3, b, x^2, and y.

2. Find the product of $\frac{x}{x^2+x-6}$, $2-\frac{x-8}{x-3}$, and x^2-9.

We have, $\frac{x}{x^2+x-6}\times\left(2-\frac{x-8}{x-3}\right)\times(x^2-9)$

$$=\frac{x}{x^2+x-6}\times\frac{2x-6-x+8}{x-3}\times(x^2-9)$$

$$=\frac{x}{\cancel{(x+3)}(x-2)}\times\frac{x+2}{\cancel{x-3}}\times\cancel{(x+3)}\cancel{(x-3)}=\frac{x(x+2)}{x-2},\ \textit{Ans.}$$

In this case, the factors cancelled are $x+3$ and $x-3$.

EXAMPLES.

Simplify the following:

3. $\frac{5a^3b^5}{14x^4y^3}\times 7xy^3$.

4. $\frac{6a^2m}{25b^2n^4}\times\frac{20b^3n^4}{3a^2m^3}$.

5. $\frac{5x}{4y}\times\frac{3y}{10z}\times\frac{8z}{9x}$.

6. $\frac{4a^6}{9b^3}\times\frac{15b^4}{7c^4}\times\frac{21c^5}{10a^5}$.

7. $\frac{3a^2b^3}{4c^4}\times\frac{6b^2c^3}{5a^6}\times\frac{10c^2a}{9b^3}$.

8. $\frac{27m^5}{20n^3x}\times\frac{15n^3}{28x^2y^3}\times\frac{7x^4}{18m^2y}$.

9. $\frac{a^2+a-30}{3a}\times\frac{5a}{a^2-4a-5}$.

10. $\frac{9m^2-1}{m^3-25m}\times\frac{m^2+5m}{3m-1}$.

11. $\frac{x^2+3x-18}{x^2-8x+12}\times\frac{2x^3-4x^2}{x^2-36}$.

12. $\frac{xy+y^2}{x^2-xy}\times\frac{x^2+xy-2y^2}{x^2+2xy+y^2}$.

13. $\frac{a^2+ab-20b^2}{a^2-ab-6b^2}\times\frac{a^2-4b^2}{a^2+5ab}$. 14. $\frac{x^4-x^2}{x^3+8}\times\frac{x^2-2x+4}{x^3+x^2+x}$.

15. $\frac{5x+15}{8x-4}\times\frac{3x-9}{10x+5}\times\frac{8x^2-2}{3x^2-27}$.

16. $\frac{a^2+2a}{a^2-3a-4}\times\frac{a^2-16}{a^2-a}\times\frac{a^2+a}{a^2+6a+8}$.

17. $\left(6-\frac{2x^2+3y^2}{x^2-y^2}\right)\left(2-\frac{3x+5y}{2x+3y}\right)$.

18. $\frac{(x+y)^2-z^2}{(x-y)^2-z^2}\times\frac{x^2-(y-z)^2}{x^2-(y+z)^2}$.

19. $\frac{a-b}{a^3+b^3}\times\frac{a^3-b^3}{a+b}\times\left(1-\frac{2ab}{a^2+ab+b^2}\right)$.

20. $\frac{a^3x+ax^3}{a^4-2a^2x^2+x^4}\times\frac{a^2+2ax+x^2}{a^2+x^2}\times\frac{a^2-2ax+x^2}{ax}$.

21. $\frac{x^4-1}{16x^4-9x^2}\times\frac{4x+3}{2x^2+2}\times\left(4x+\frac{x}{x-1}\right)$.

DIVISION OF FRACTIONS.

145. Required the quotient of $\frac{a}{b}$ divided by $\frac{c}{d}$.

Let $\frac{a}{b}\div\frac{c}{d}=x$. (1)

Then since the dividend is the product of the divisor and quotient (§ 54), we have

$$\frac{a}{b}=\frac{c}{d}\times x.$$

Multiplying each of these equals by $\frac{d}{c}$ (§ 9, 1), we have

$$\frac{a}{b}\times\frac{d}{c}=\frac{c}{d}\times x\times\frac{d}{c}=x. \quad (2)$$

From (1) and (2), $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}$. (§ 9, 4)

Therefore, *to divide one fraction by another, multiply the dividend by the divisor inverted.*

Integral or mixed expressions should be expressed in a fractional form (§§ 135, 138) before applying the rule.

146. 1. Divide $\frac{6a^2b}{5x^3y^4}$ by $\frac{9a^2b^3}{10x^2y^7}$.

We have, $\frac{6a^2b}{5x^3y^4} \div \frac{9a^2b^3}{10x^2y^7} = \frac{6a^2b}{5x^3y^4} \times \frac{10x^2y^7}{9a^2b^3} = \frac{4y^3}{3b^2x}$, *Ans.*

2. Divide $9 + \frac{5y^2}{x^2 - y^2}$ by $3 + \frac{5y}{x - y}$.

We have,

$$\left(9 + \frac{5y^2}{x^2 - y^2}\right) \div \left(3 + \frac{5y}{x - y}\right) = \frac{9x^2 - 9y^2 + 5y^2}{x^2 - y^2} \div \frac{3x - 3y + 5y}{x - y}$$

$$= \frac{9x^2 - 4y^2}{x^2 - y^2} \times \frac{x - y}{3x + 2y} = \frac{(3x + 2y)(3x - 2y)}{(x + y)(x - y)} \times \frac{x - y}{3x + 2y}$$

$$= \frac{3x - 2y}{x + y}, \textit{ Ans.}$$

EXAMPLES.

Simplify the following:

3. $\frac{24a^5b}{7x^4y^3} \div 8a^3b^2$.

4. $\frac{21an^5}{10b^4m} \div \frac{14a^5n^2}{15b^4m^3}$.

5. $\frac{3}{x^2 - 6x + 8} \div \frac{2}{x^2 - x - 12}$.

6. $\frac{4m^2 - 25n^2}{16m^2 - 9n^2} \div \frac{2mn - 5n^2}{4m^2 + 3mn}$.

7. $\left(\frac{a^2}{5b} - \frac{a}{2}\right) \div \left(\frac{2a^2}{3b} + \frac{a}{2}\right)$.

8. $\frac{a^2 + 10a + 21}{a^3 - 4a^2 + 3a} \div \frac{a^2 - 9}{a^3 - a^2}$.

9. $\frac{x^2 + 4xy + 4y^2}{x - y} \div \frac{xy + 2y^2}{x^2 - xy}$.

10. $\frac{x^2 - x}{x^3 + 1} \div \frac{x^2 - 2x + 1}{x^3 - x^2 + x}$.

11. $\frac{a^3 - 8}{a^2 + 7a + 10} \div \frac{a^2 + 2a + 4}{a^2 + 2a}$.

12. $\frac{a^2 - 5ab - 14b^2}{a^2 + 5ab - 24b^2} \div \frac{a^2 - 3ab - 28b^2}{a^2 - 8ab + 15b^2}$.

13. $\left(5-\frac{a^2-19x^2}{a^2-4x^2}\right)\div\left(3-\frac{a-5x}{a-2x}\right).$

14. $\frac{a^2-b^2-c^2-2bc}{a^2-b^2-c^2+2bc}\div\frac{a-b-c}{a+b-c}.$

COMPLEX FRACTIONS.

147. A **Complex Fraction** is a fraction having one or more fractions in either or both of its terms.

It is simply a case in division of fractions, its numerator being the dividend, and its denominator the divisor.

148. 1. Simplify $\frac{a}{b-\frac{c}{d}}$.

We have, $\frac{a}{b-\frac{c}{d}}=\frac{a}{\frac{bd-c}{d}}=a\times\frac{d}{bd-c}$ (§ 146) $=\frac{ad}{bd-c}$, *Ans.*

It is often advantageous to simplify a complex fraction by multiplying its numerator and denominator by the lowest common multiple of their denominators (§ 129).

2. Simplify $\frac{\frac{a}{a-b}-\frac{a}{a+b}}{\frac{b}{a-b}+\frac{a}{a+b}}$.

The L. C. M. of $a+b$ and $a-b$ is $(a+b)(a-b)$.

Multiplying both terms by $(a+b)(a-b)$, we have

$$\frac{a(a+b)-a(a-b)}{b(a+b)+a(a-b)}=\frac{a^2+ab-a^2+ab}{ab+b^2+a^2-ab}=\frac{2ab}{a^2+b^2}, \textit{ Ans.}$$

EXAMPLES.

Simplify the following:

3. $\frac{\frac{a}{b}-\frac{c}{d}}{\frac{a}{b}+\frac{c}{d}}.$

4. $\frac{1+\frac{1}{2m}}{m-\frac{1}{4m}}.$

5. $\frac{x-\frac{1}{x^2}}{1-\frac{1}{x}}.$

6. $\dfrac{\dfrac{m}{n}-\dfrac{n}{m}}{\dfrac{(m+n)^2}{mn}-4}.$

7. $\dfrac{\dfrac{2x}{3y}-2+\dfrac{3y}{2x}}{\dfrac{2}{y}-\dfrac{3}{x}}.$

8. $\dfrac{a-\dfrac{a-x}{1+ax}}{1+\dfrac{a^2-ax}{1+ax}}.$

9. $\dfrac{\dfrac{x}{y}+1-\dfrac{20y}{x}}{\dfrac{x}{y}-2-\dfrac{8y}{x}}.$

10. $\dfrac{\dfrac{a}{b}-\dfrac{a-b}{a+b}}{\dfrac{b}{a}+\dfrac{a-b}{a+b}}.$

11. $\dfrac{\dfrac{x^2}{y^2}+\dfrac{8y}{x}}{\dfrac{x}{y}-2+\dfrac{4y}{x}}.$

12. $\dfrac{x^2-13+\dfrac{36}{x^2}}{x+1-\dfrac{6}{x}}.$

13. $\dfrac{\dfrac{x}{x+y}-\dfrac{x-y}{x}}{\dfrac{x}{x-y}-\dfrac{x+y}{x}}.$

14. $\dfrac{a-\dfrac{2b^2}{a-b}}{a+\dfrac{b^2}{a+2b}}.$

15. $\dfrac{a^2+a+1+\dfrac{2}{a-1}}{a+\dfrac{1}{a-1}}.$

16. $\dfrac{\dfrac{2a+b}{a+3b}+\dfrac{2a-b}{a-3b}}{\dfrac{2a+b}{a-3b}-\dfrac{2a-b}{a+3b}}.$

17. Simplify $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}.$

We have, $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}=\dfrac{1}{1+\dfrac{x}{x+1}}=\dfrac{x+1}{x+1+x}=\dfrac{x+1}{2x+1}$, *Ans.*

In examples like the above, begin by simplifying the *lowest* complex fraction; first multiply both terms of $\dfrac{1}{1+\dfrac{1}{x}}$ by x, giving $\dfrac{x}{x+1}$, and then multiply both terms of $\dfrac{1}{1+\dfrac{x}{x+1}}$ by $x+1$, giving $\dfrac{x+1}{x+1+x}$.

Simplify the following:

18. $3-\dfrac{2}{5+\dfrac{4}{7+\dfrac{6}{x}}}.$

19. $1-\dfrac{1}{2-\dfrac{1}{3-\dfrac{a}{1-a}}}.$

20. $$\frac{1-\frac{8(a^2+b^2)}{9a^2-b^2}}{1-\frac{2(a+2b)}{3a+b}}.$$

23. $$\frac{\frac{x+y}{x-y}-\frac{x^2+y^2}{x^2-y^2}}{\frac{x^2}{x+y}-\frac{x(x^2+y^2)}{(x+y)^2}}.$$

21. $$\frac{\frac{x+a}{x-a}-\frac{x-a}{x+a}}{\frac{x^2+a^2}{(x-a)^2}-1}.$$

24. $$\frac{m-n-\frac{2n(m-n)}{m+n}}{\frac{m^2+n^2}{mn+n^2}-1}.$$

22. $$\frac{\frac{1-x^2}{1+x^2}-\frac{1+x^2}{1-x^2}}{\frac{1-x}{1+x}-\frac{1+x}{1-x}}.$$

25. $$\frac{\frac{a+b}{a-b}-\frac{a^3+b^3}{a^3-b^3}}{\frac{a+b}{a-b}+\frac{a^3+b^3}{a^3-b^3}}.$$

MISCELLANEOUS AND REVIEW EXAMPLES.

149. Reduce each of the following to a fractional form:

1. $9a-2-\frac{(6a-5)^2}{4a+3}.$ 2. $\frac{(a^2+b^2)^2}{a^2-ab+b^2}-(a^2+ab+b^2)$

Simplify the following:

3. $$\frac{1}{2a-3x}-\frac{2a}{(2a-3x)^2}+\frac{6ax}{(2a-3x)^3}.$$

4. $$\frac{(1-x)(1-y)}{(1+xy)^2-(x+y)^2}.$$

5. $$\frac{(a^2-2)^2-a^2}{a^4-3a^2-4}.$$

6. $$\left(\frac{1}{3y^2}-\frac{1}{xy}+\frac{3}{4x^2}\right)\div\left(\frac{1}{3y}-\frac{1}{2x}\right).$$

7. $$\left(\frac{a^2}{b^2}+1+\frac{b^2}{a^2}\right)\left(\frac{a}{b}-\frac{b}{a}\right).$$

8. $$\frac{a+b}{a-b}+\frac{c+d}{c-d}+\frac{2(ac-bd)}{(b-a)(c-d)}.$$

9. $$\frac{6xy-(x+2y)^2}{x^3+8y^3}.$$

10. $$\frac{(x^2-6x-4)^2-144}{(x^2+x-11)^2-81}.$$

11. $\dfrac{a}{a^2-3a+2}+\dfrac{3a}{a^2-7a+10}-\dfrac{4a}{a^2-6a+5}.$

12. $\left(2a+3-\dfrac{24a}{2a+3}\right)\left(2a-3+\dfrac{24a}{2a-3}\right).$

13. $\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^3-y^3}{x^3+y^3}.$

14. $\dfrac{a^5+a^4b-a^2b^3-ab^4}{a^4b-a^3b^2+ab^4-b^5}.$

15. $\left(\dfrac{x^3}{y^3}-\dfrac{x}{y}+\dfrac{y}{x}-\dfrac{y^3}{x^3}\right)\left(\dfrac{x}{y}+\dfrac{y}{x}\right).$

16. $\dfrac{a^2cd+abd^2-abc^2-b^2cd}{a^2cd-abd^2-abc^2+b^2cd}.$

17. $\left(\dfrac{x}{x-2}+\dfrac{5}{x-8}\right)\left(\dfrac{x-3}{3x-8}-\dfrac{2}{x+2}\right).$

18. $\left(\dfrac{2}{x^2+3x+2}-\dfrac{1}{x+1}\right)\div\left(\dfrac{1}{x^2+3x+2}-\dfrac{1}{x+2}\right).$

19. $\dfrac{m}{2(m-n)}-\dfrac{m}{2(m+n)}+\dfrac{2m^4}{n^2(m^2-n^2)}.$

20. $\dfrac{(a+b+c)^2-(a+b-c)^2}{(a-b+c)^2-(a-b-c)^2}.$

21. $\dfrac{2a(2a-3bx)+3b(3b+2ax)}{(2a-3bx)^2+(3b+2ax)^2}.$

22. $\dfrac{x-y}{x+z}+\dfrac{x+z}{x-y}-\dfrac{(y+z)^2}{(x-y)(x+z)}.$

23. $\dfrac{a-b-\dfrac{a^2b-ab^2}{(a+b)^2}}{\dfrac{a+b}{a-b}+\dfrac{a^2+b^2}{a^2-b^2}}.$

24. $\dfrac{4x^3-16x^2+17x-3}{6x^3-17x^2+8x+6}.$

25. $\dfrac{1}{a}\left(\dfrac{1}{x-a}+\dfrac{1}{x+2a}\right)-\dfrac{3}{x^2+ax-2a^2}.$

26. $\dfrac{a^2-6ab+9b^2}{a^2-4ab+4b^2}\div\left(\dfrac{a^2-9b^2}{a^2-4b^2}\times\dfrac{a^2-ab-6b^2}{a^2+ab-6b^2}\right).$

27. $\dfrac{x^4-2x^2y^2+y^4}{x^4+2x^2y^2+y^4}\div\left(\dfrac{x^2-y^2}{xy}\div\dfrac{(x+y)^2-2xy}{xy}\right).$

28. $\left[\left(\frac{x}{x-1}-\frac{1}{x+1}\right)\div\left(\frac{x}{x-1}+\frac{1}{x+1}\right)\right](x^2+2x-1).$

29. $\frac{m^2+2mn+4n^2}{2m-3n}\times\frac{4m^2-9n^2}{m^3-8n^3}\div\frac{2m+3n}{m^2+4n^2}.$

30. $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-c)(b-a)}.$ 31. $\dfrac{\frac{x+y}{x-y}-\frac{x^3-y^3}{x^3+y^3}}{\frac{x^3+y^3}{x^3-y^3}-\frac{x-y}{x+y}}.$

32. $\frac{x+z}{(x-y)(y-z)}-\frac{x+y}{(y-z)(z-x)}+\frac{y+z}{(z-x)(x-y)}.$

33. $\left(1-\frac{ab}{a^2-ab+b^2}\right)\left(1-\frac{ab}{a^2+2ab+b^2}\right)\times\frac{a^3+b^3}{a^3-b^3}.$

34. $\frac{1}{1-x}+\frac{1}{1+x}+\frac{2}{1+x^2}+\frac{4}{1+x^4}.$

(First add the first two fractions; to the result add the third fraction, and to this result add the fourth fraction.)

35. $\frac{a}{a-2}+\frac{a}{a+2}+\frac{2a^2}{a^2+4}+\frac{4a^4}{a^4+16}.$

36. $\frac{1}{x-1}-\frac{1}{x+1}+\frac{1}{x-2}-\frac{1}{x+2}.$

(First combine the first two fractions, then the last two, and then add these results.)

37. $\frac{1}{a-b}-\frac{1}{a+b}+\frac{2a}{a^2-b^2}-\frac{2a}{a^2+b^2}.$

38. $\frac{1}{x-1}-\frac{1}{x+1}+\frac{3x^2}{x^3+1}-\frac{3x^2}{x^3-1}.$

39. $\frac{4x-3}{6x^2+13x-5}+\frac{2x-5}{12x^2+5x-3}.$

(Find the L. C. M. of the denominators by the method of § 126.)

40. $\frac{3a+2}{6a^2-a-12}-\frac{5a-1}{10a^2-19a+6}.$

XIII. SIMPLE EQUATIONS (Continued).

SOLUTION OF EQUATIONS CONTAINING FRACTIONS.

150. Clearing of Fractions.

Consider the equation $\frac{2x}{3}-\frac{5}{4}=\frac{5x}{6}-\frac{9}{8}$.

The lowest common multiple of 3, 4, 6, and 8 is 24.

Multiplying each term of the equation by 24 (§ 71, 2), we have

$$16x-30=20x-27,$$

where the denominators have been removed.

We derive from the above the following rule for clearing an equation of fractions:

Multiply each term by the lowest common multiple of the given denominators.

151. 1. Solve the equation $\frac{7x}{6}-\frac{5}{3}=\frac{3x}{5}-\frac{1}{4}$.

The L. C. M. of 6, 3, 5, and 4 is 60.

Multiplying each term of the equation by 60, we have

$$70x-100=36x-15.$$

Transposing, $70x-36x=100-15.$

Uniting terms, $34x=85.$

Dividing by 34, $x=\frac{85}{34}=\frac{5}{2}$, *Ans.*

EXAMPLES.

Solve the following equations:

2. $x+\frac{x}{2}-\frac{3x}{5}=9.$

3. $\frac{5x}{3}-\frac{3x}{4}+\frac{11}{6}=0.$

4. $\frac{3x}{2}-\frac{x}{3}=\frac{5x}{4}+\frac{1}{8}.$

5. $\frac{4x}{9}-\frac{2}{3}=\frac{5x}{6}-\frac{3x}{2}.$

6. $\frac{7x}{2}-\frac{4x}{3}+\frac{2x}{5}=-\frac{11}{6}.$

7. $\frac{3}{5x}-1=\frac{7}{10}-\frac{1}{4x}.$

8. $\frac{5}{18x}-\frac{1}{6x}=\frac{1}{4}-\frac{8}{9x}.$

9. $\frac{3x}{2}+\frac{5}{14}-\frac{x}{3}=\frac{7x}{6}-\frac{4x}{7}.$

10. $\frac{2x}{5}-\frac{x}{2}+\frac{3}{10}=\frac{7x}{8}-\frac{3x}{4}.$

11. $\frac{2}{3x}-\frac{3}{4x}-\frac{4}{5x}+\frac{5}{6x}=\frac{1}{20}.$

If a fraction whose numerator is a polynomial is preceded by a $-$ sign, it is convenient, on clearing of fractions, to enclose the numerator in a parenthesis, as shown in Ex. 12.

If this is not done, care must be taken to *change the sign of each term of the numerator* when the denominator is removed.

12. Solve the equation $\frac{3x-1}{4}-\frac{4x-5}{5}=4+\frac{7x+5}{10}.$

The L. C. M. of 4, 5, and 10 is 20; multiplying each term by 20, we have

$$15x-5-(16x-20)=80+14x+10.$$

Whence, $15x-5-16x+20=80+14x+10.$

Transposing, $15x-16x-14x=80+10+5-20.$

Uniting terms, $-15x=75.$

Dividing by -15, $x=-5$, *Ans.*

Solve the following equations:

13. $4x+\frac{8x-12}{7}=\frac{9x}{2}.$

14. $\frac{5x}{3}-\frac{2x-2}{9}=x-2.$

15. $2x-\frac{3x+7}{11}=\frac{x}{2}+1.$

16. $x-\frac{3x+7}{3}=\frac{8x-4}{7}-1.$

17. $\frac{2x-5}{7}-\frac{3x-8}{6}=-\frac{2}{3}.$

18. $\frac{x+2}{10}=\frac{9}{35}-\frac{3x+14}{14}.$

19. $\frac{11x+4}{2}-\frac{14x+3}{4}-\frac{10x+7}{8}=0.$

20. $\frac{3x+4}{2}-\frac{1}{8}(19x-3)=1-\frac{7x+8}{12}.$

21. $\frac{1}{5}(x+16)-\frac{3x-1}{20}=\frac{21}{5}-\frac{x+3}{2}.$

22. $\frac{10(x+2)}{9}-\frac{5x-4}{12}-\frac{5x+12}{6}=1\frac{1}{9}.$

23. $\frac{14x-1}{2}-\frac{1}{3}(8x-5)+\frac{1}{6}(10x-7)=\frac{3(4x+1)}{4}.$

24. $\frac{11x-2}{3}-\frac{1}{2}(3x-1)=\frac{17x+7}{6}-\frac{2}{9}(7x-2).$

25. $\frac{2x+4}{5}+\frac{7x-1}{2}=\frac{13x+5}{3}-\frac{11x-3}{10}.$

26. $\frac{7x-8}{14}-\frac{7x+6}{4x}=\frac{x-5}{2}-\frac{4x+9}{7x}.$

27. $\frac{3(x-3)}{2}-\frac{2(x^2-5)}{3x}-\frac{5x^3-12}{6x^2}=-\frac{9}{2}.$

28. Solve the equation $\frac{2}{x-2}-\frac{5}{x+2}-\frac{2}{x^2-4}=0.$

The L. C. M. of $x-2$, $x+2$, and x^2-4 is x^2-4.

Multiplying each term by x^2-4, we have

$$2(x+2)-5(x-2)-2=0.$$

Whence, $\quad 2x+4-5x+10-2=0.$

Transposing, $\quad 2x-5x=-4-10+2.$

Uniting terms, $\quad -3x=-12.$

Dividing by -3, $\quad x=4$, *Ans.*

If the denominators are partly monomial and partly polynomial, it is often advantageous to clear of fractions at first partially; multiplying each term of the equation by the L. C. M. of the *monomial* denominators.

29. Solve the equation $\frac{6x+1}{15}-\frac{2x-4}{7x-16}=\frac{2x-1}{5}.$

The L.C.M. of the monomial denominators is 15.
Multiplying each term by 15, we have

$$6x+1-\frac{30x-60}{7x-16}=6x-3.$$

Transposing and uniting terms, $4=\frac{30x-60}{7x-16}.$

Multiplying by $7x-16$, $28x-64=30x-60.$

Transposing, $28x-30x=64-60.$

Uniting terms, $-2x=4.$

Dividing by -2, $x=-2$, *Ans.*

Solve the following equations:

30. $\frac{9}{5x+2}-\frac{7}{3x+4}=0.$

31. $\frac{2x+3}{3x-4}=\frac{4x+5}{6x-1}.$

32. $\frac{15x^2-5x-8}{3x^2+6x+4}=5.$

33. $\frac{6x+5}{2x(x-1)}=\frac{3x-2}{x^2-1}.$

34. $\frac{3x}{2x+3}-\frac{2x}{2x-3}=\frac{2x^2-5}{4x^2-9}.$

35. $\frac{7x^2+10x-24}{(x+1)^2}-\frac{5}{x+1}=7.$

36. $\frac{6x+7}{6}=x+2-\frac{7x-13}{2(2x+1)}.$

37. $\frac{x}{3}+\frac{2x^2+7}{3x-4}=\frac{9x-2}{9}.$

38. $\frac{9}{3x-5}-\frac{2}{x-2}=\frac{1}{x-3}.$

39. $\frac{2x+7}{14}-\frac{5x-4}{3x+1}=\frac{x+6}{7}.$

40. $\frac{3}{x-2}-\frac{4}{2x-1}=\frac{3}{3x+2}.$

41. $\frac{2(x-7)}{x^2+3x-28}+\frac{x-2}{x-4}-\frac{x+3}{x+7}=0.$

42. $\frac{5x+1}{5}-\frac{4x+7}{6x+11}-\frac{3x-2}{3}=0.$

43. $\frac{1}{2x+3}+\frac{3}{3x-2}=\frac{6}{4x+1}.$

44. $\frac{2x+1}{2x-16}-\frac{2x-1}{2x+12}=\frac{9x+17}{x^2-2x-48}.$

45. $\frac{x-2}{x+2}-\frac{x-1}{x+1}+\frac{2x+4}{x^2-1}=0.$

46. $\frac{2+3x}{3-x}-\frac{2-3x}{3+x}=\frac{36-4x}{x^2-9}.$

47. $\frac{2x^2+3x-1}{2x+1}+\frac{2x^2-3x+1}{2x-1}=2x.$

48. $\frac{(x+1)(x+3)}{(x+5)(x+7)}=\frac{x-6}{x+2}.$

49. $\frac{3x^2+5x-4}{4x^2-3x+2}=\frac{3x+5}{4x-3}.$

50. $\frac{7(3x-8)}{3(x-3)}+\frac{3(x-2)}{3x-1}=8.$

51. $\frac{2x+7}{6x-4}-\frac{3x-5}{9x+6}=\frac{17x+2}{9x^2-4}.$

52. $\frac{x-2}{x-3}-\frac{x-3}{x-4}=\frac{x-5}{x-6}-\frac{x-6}{x-7}.$

(First combine the fractions in the first member; then the fractions in the second member.)

53. $\frac{3x+5}{7}-\frac{3x-2}{14}=\frac{6x-5}{28}-\frac{7x+3}{4(4x-3)}.$

SOLUTION OF LITERAL EQUATIONS.

152. A **Literal Equation** is one in which some or all of the known quantities are represented by letters; as,

$$2x+a=bx^2-10.$$

153. 1. Solve the equation $(b-cx)^2-(a-cx)^2=b(b-a)$.

Performing the operations indicated, we have

$$b^2-2bcx+c^2x^2-(a^2-2acx+c^2x^2)=b^2-ab.$$

Whence, $b^2-2bcx+c^2x^2-a^2+2acx-c^2x^2=b^2-ab.$

Transposing and uniting terms, $2acx-2bcx=a^2-ab.$

Factoring both members, $2cx(a-b)=a(a-b).$

Dividing by $2c(a-b)$, $x=\frac{a(a-b)}{2c(a-b)}=\frac{a}{2c}$, *Ans.*

EXAMPLES.

Solve the following equations:

2. $a(3bx - 2a) = b(2a - 3bx).$

3. $(x + a)^2 + (b + c)^2 = (x - a)^2 + (b - c)^2.$

4. $\dfrac{x - a}{x} + \dfrac{2x}{x - a} = 3.$ 6. $\dfrac{x^2 - b}{ax} - \dfrac{b - x}{a} = \dfrac{2x}{a} - \dfrac{b}{x}.$

5. $\dfrac{3x - 4}{3x + 4} = \dfrac{5m - 2n}{5m + 2n}.$ 7. $x - 1 - \dfrac{x - 2}{m} = \dfrac{1}{m^2}.$

8. $\dfrac{5x - 2a}{2a} - \dfrac{9x - 5a^3}{3a^3} + \dfrac{3(x + 2a^3)}{a^3} + \dfrac{5x}{6a} = 0.$

9. $\dfrac{ax - b}{bx} + \dfrac{bx + a}{ax} = 2 + \dfrac{a - b}{abx}.$

10. $2(x - b)(2a - 3b - 3x) - (2a - 3x)(b + 2x) = 0.$

11. $(x + m)(x + n) - (x - m)(x - n) = 2(m + n)^2.$

12. $\dfrac{x - b}{x - 2a} - \dfrac{x + b}{x + 2a} = \dfrac{4a^2 - b^2}{x^2 - 4a^2}.$

13. $\dfrac{2x + 3a}{2x - 3b} = \dfrac{3x + 4b}{3x - 4a}.$ 14. $\dfrac{2nx - 3}{nx - 1} = 5 - \dfrac{9nx + 2}{3nx - 1}.$

15. $\dfrac{a - b}{x - c} + \dfrac{b - c}{x - a} + \dfrac{c - a}{x} = 0.$

16. $(x + a)^3 + (x - a)^3 = 2x(x^2 - a^2) - 24a^3.$

17. $\dfrac{3x(a - b)}{x^2 - b^2} - \dfrac{a - 2b}{x + b} + \dfrac{a - b}{b - x} = 0.$

18. $\dfrac{x}{2} - \dfrac{a - 2bcx}{4bc} = \dfrac{5x}{6c} - \dfrac{8ac - 8bx - 9a}{12bc}.$

19. $(x - 2a + 3b)^2 - (x - 2a)(x + 3b) - 6ab = 0.$

20. $\frac{a}{x-a}-\frac{b}{x-b}=\frac{b^2-a^2}{b^2-bx}$.

22. $\frac{(2x-3m)^2}{(2x-3n)^2}=\frac{x-3m}{x-3n}$.

21. $\frac{ax}{x+b}+\frac{bx}{x+a}=a+b$.

23. $\frac{2(a+b)}{x}=\frac{x+b}{x-b}-\frac{x-a}{x+a}$.

24. $\frac{1}{x-a}+\frac{1}{x-b}=\frac{2x-a-b}{x(x-a-b)}$.

25. $\frac{x+4a+b}{x+a+b}+\frac{4x+a+2b}{x+a-b}=5$.

SOLUTION OF EQUATIONS INVOLVING DECIMALS.

154. 1. Solve the equation $.17x-.23=.113x+.112$.

Transposing, $.17x-.113x=.23+.112$.

Uniting terms, $.057x=.342$.

Dividing by .057, $x=\frac{.342}{.057}=6$, *Ans.*

EXAMPLES.

Solve the following equations:

2. $2.9x-1.98=1.4x-1.845$.

3. $.05x+.117=.186x-.2x-.139$.

4. $.6x-.265+.03=.4+.66x-.187x$.

5. $.4(1.7x-.6)=.95x+5.16$.

6. $.08(35x-2.3)=.9(7x+.18)-.997$.

7. $2.8x-\frac{.29x+.0184}{.7}=.5x-.064$.

8. $3.89-\frac{.4x+.708}{2x}=\frac{18}{5}-\frac{.3}{x}$.

9. $\frac{.7x+.371}{.9}-\frac{.3x-.256}{.6}=.45$.

10. $\frac{2-3x}{1.5}-\frac{3x-14}{9}=\frac{x-2}{1.8}-\frac{10x-9}{2.25}$.

PROBLEMS.

155. 1. Divide 43 into two parts such that three-eighths of one part may equal two-ninths of the other.

Let $x =$ one part.

Then, $43 - x =$ the other.

By the conditions, $\frac{3x}{8} = \frac{2}{9}(43 - x)$.

Clearing of fractions, $27x = 16 \times 43 - 16x$.

Transposing, $43x = 16 \times 43$.

Dividing by 43, $x = 16$, one part.

Whence, $43 - x = 27$, the other part.

2. The fifth part of a number exceeds its eighth part by 3; what is the number?

3. What number is that from which if four-sevenths of itself be subtracted, the result will equal three-fourths of the number diminished by 18?

4. What number exceeds the sum of its third, sixth, and fourteenth parts by 18?

5. Divide 45 into two parts such that the sum of four-ninths the greater and two-thirds the less shall equal 24.

6. Divide 56 into two parts such that five-eighths the greater shall exceed seven-twelfths the less by 6.

7. Divide $124 between A, B, and C so that A's share may be five-sixths of B's, and C's nine-tenths of A's.

8. A man travelled 768 miles. He went four-fifths as many miles by water as by rail, and five-twelfths as many by carriage as by water. How many miles did he travel in each manner?

9. A's age is three-eighths of B's, and eight years ago it was two-sevenths of B's age; find their ages at present.

10. A has \$52, and B \$38. After giving B a certain sum, A has only three-sevenths as much money as B. What sum was given to B?

11. I paid a certain sum for a picture, and the same price for a frame. If the picture had cost \$4 more, and the frame 30 cents less, the price of the frame would have been one-third that of the picture. Find the cost of the picture.

12. A can do a piece of work in 8 days which B can perform in 10 days. In how many days can it be done by both working together?

Let $x =$ the number of days required.

Then, $\frac{1}{x} =$ the part both can do in one day.

Also, $\frac{1}{8} =$ the part A can do in one day,

and $\frac{1}{10} =$ the part B can do in one day.

By the conditions, $\frac{1}{8} + \frac{1}{10} = \frac{1}{x}.$

$$5x + 4x = 40.$$

$$9x = 40.$$

Whence, $x = 4\frac{4}{9}$, the number of days required.

13. The second digit of a number exceeds the first by 2; and if the number, increased by 6, be divided by the sum of its digits, the quotient is 5. Find the number.

Let $x =$ the first digit.

Then, $x + 2 =$ the second digit,

and $2x + 2 =$ the sum of the digits.

The number itself is equal to 10 times the first digit, plus the second.

Then, $10x + (x + 2)$, or $11x + 2 =$ the number.

By the conditions, $\frac{11x + 2 + 6}{2x + 2} = 5.$

$$11x + 8 = 10x + 10.$$

Whence, $x = 2.$

Then, $11x + 2 = 24$, the number required.

sum of parts actually worked is equal
to the whole amount of work

14. A can do a piece of work in 18 days, and B can do the same in 24 days. In how many days can it be done by both working together?

15. A can do a piece of work in $3\frac{1}{3}$ hours which B can do in $3\frac{3}{4}$ hours, and C in $3\frac{3}{5}$ hours. In how many hours can it be done by all working together?

16. A tank can be filled by one pipe in 9 hours, and emptied by another in 21 hours. In what time will the tank be filled if both pipes be opened?

17. A vessel can be filled by three taps; by the first alone in $7\frac{1}{2}$ minutes, by the second alone in $4\frac{1}{5}$ minutes, and by the third alone in $4\frac{3}{8}$ minutes. In what time will it be filled if all the taps be opened?

18. The first digit of a number is 4 less than the second; and if the number be divided by the sum of its digits, the quotient is 4. Find the number.

19. The second digit of a number is one-fourth of the first; and if the number, diminished by 10, be divided by the difference of its digits, the quotient is 12. Find the number.

20. If a certain number be diminished by 23, one-fourth of the result is as much below 37 as the number itself is above 56. Find the number.

21. What number is that, seven-eighths of which is as much below 21 as three-tenths of it exceeds $2\frac{1}{2}$?

22. B is 24 years older than A; and when A is twice his present age, B will be $\frac{3}{2}$ as old as he now is. How old is each?

23. The denominator of a fraction exceeds the numerator by 5. If the denominator be decreased by 20, the resulting fraction, increased by 1, is equal to twice the original fraction. Find the fraction.

24. Divide 44 into two parts such that one divided by the other shall give 2 as a quotient and 5 as a remainder.

Let $x =$ the divisor.

Then, $44 - x =$ the dividend.

Now since the dividend is equal to the product of the divisor and quotient, plus the remainder, we have

$$44 - x = 2x + 5.$$

$$-3x = -39.$$

Whence, $x = 13$, the divisor,

and $44 - x = 31$, the dividend.

25. Two persons, A and B, 63 miles apart, start at the same time and travel towards each other. A travels at the rate of 4 miles an hour, and B at the rate of 3 miles an hour. How far will each have travelled when they meet?

Let $4x =$ the number of miles that A travels.

Then, $3x =$ the number of miles that B travels.

By the conditions,

$$4x + 3x = 63.$$

$$7x = 63.$$

$$x = 9.$$

Whence, $4x = 36$, the number of miles that A travels,

and $3x = 27$, the number of miles that B travels.

Note. It is often advantageous, as in Ex. 25, to represent the unknown quantity by some *multiple* of x instead of by x itself.

26. Divide 49 into two parts such that one divided by the other may give 2 as a quotient and 7 as a remainder.

27. Two men, A and B, 66 miles apart, set out at the same time and travel towards each other. A travels at the rate of 15 miles in 4 hours, and B at the rate of 9 miles in 2 hours. How far will each have travelled when they meet?

28. Divide 134 into two parts such that one divided by the other may give 3 as a quotient and 26 as a remainder.

29. The denominator of a fraction is 7 less than the numerator; and if 5 be added to the numerator, the value of the fraction is $\frac{8}{5}$. Find the fraction.

30. The second digit of a number exceeds the first by 4; and if the number, increased by 39, be divided by the sum of its digits, the quotient is 7. Find the number.

31. I paid a certain sum for a horse, and seven-tenths as much for a carriage. If the horse had cost $70 less, and the carriage $50 more, the price of the horse would have been four-fifths that of the carriage. What was the cost of each?

32. A can do a piece of work in 15 hours, which B can do in 25 hours. After A has worked for a certain time, B completes the job, working 9 hours longer than A. How many hours did A work?

33. A man owns a horse, a carriage worth $100 more than the horse, and a harness. The horse and harness are together worth three-fourths the value of the carriage, and the carriage and harness are together worth $50 less than twice the value of the horse. Find the value of each.

34. The rate of an express train is $\frac{3}{2}$ that of a slow train, and it covers 180 miles in two hours less time than the slow train. Find the rate of each train.

35. Two men, A and B, 57 miles apart, set out, B 20 minutes after A, and travel towards each other. A travels at the rate of 6 miles an hour, and B at the rate of 5 miles an hour. How far will each have travelled when they meet?

36. A grocer buys eggs at the rate of 4 for 7 cents. He sells one-fourth of them at the rate of 5 for 12 cents, and the remainder at the rate of 6 for 11 cents, and makes 27 cents by the transaction. How many eggs did he buy?

37. At what time between 3 and 4 o'clock are the hands of a watch opposite to each other?

Let $x =$ the number of minute-spaces passed over by the minute-hand from 3 o'clock to the required time.

Then, since the hour-hand is 15 minute-spaces in advance of the minute-hand at 3 o'clock, $x - 15 - 30$, or $x - 45$, will represent the number of minute-spaces passed over by the hour-hand.

But the minute-hand moves 12 times as fast as the hour-hand.

Whence, $$x = 12(x - 45).$$

$$x = 12x - 540.$$

$$-11x = -540$$

$$x = 49\tfrac{1}{11}.$$

Then the required time is $49\frac{1}{11}$ minutes after 3 o'clock.

38. At what time between 1 and 2 are the hands of a watch opposite to each other?

39. At what time between 6 and 7 is the minute-hand of a watch 5 minutes in advance of the hour-hand?

40. At what time between 4 and 5 are the hands of a watch together?

41. At what time between 5 and 5.30 are the hands of a watch at right angles to each other?

42. The sum of the digits of a number is 15; and if the number be divided by its second digit, the quotient is 12, and the remainder 3. Find the number.

43. A man has 11 hours at his disposal. How far can he ride in a coach which travels $4\frac{1}{2}$ miles an hour, so as to return in time, walking back at the rate of $3\frac{3}{4}$ miles an hour?

44. A, B, and C together can do a piece of work in $1\frac{3}{5}$ days; B's work is one-half of A's, and C's three-fourths of B's. How many days will it take each working alone?

45. At what time between 9 and 10 are the hands of a watch together?

46. A, B, C, and D found a sum of money. They agreed that A should receive $4 less than one-third, B $2 more than one-fourth, C $3 more than one-fifth, and D the remainder, $25. How much did A, B, and C receive?

47. At what time between 8 and 9 are the hands of a watch opposite to each other?

48. A vessel can be emptied by three taps; by the first alone in 90 minutes, by the second alone in 144 minutes, and by the third alone in 4 hours. In what time will it be emptied if all the taps be opened?

49. A and B start in business, B putting in $\frac{3}{4}$ as much capital as A. The first year, A loses $500, and B gains $\frac{1}{5}$ of his money; the second year, A gains $\frac{1}{4}$ of his money, and B loses $205; and they have now equal amounts. How much had each at first?

50. A man buys two pieces of cloth, one of which contains 6 yards more than the other. For the larger he pays at the rate of $7 for 10 yards, and for the smaller at the rate of $5 for 3 yards. He sells the whole at the rate of 9 yards for $11, and makes $5 on the transaction. How many yards were there in each piece?

51. A man loaned a certain sum for 3 years at 5 per cent compound interest; that is, at the end of each year there was added $\frac{1}{20}$ to the sum due. At the end of the third year, there was due him $2130.03. Find the amount loaned.

52. At what times between 7 and 8 are the hands of a watch at right angles to each other?

53. At what time between 2 and 3 is the hour-hand of a watch one minute in advance of the minute-hand?

54. Gold is $19\frac{1}{4}$ times as heavy as water, and silver $10\frac{1}{2}$ times. A mixed mass weighs 1960 oz., and displaces 120 oz. of water. How many ounces of each metal does it contain?

55. A merchant increases his capital annually by one-third of it, and at the end of each year takes out \$ 1800 for expenses. At the end of three years, after taking out his expenses, he finds that his capital is \$ 3800. What was his capital at first?

56. A and B together can do a piece of work in $2\frac{2}{9}$ days, B and C in $2\frac{8}{11}$ days, and C and A in $2\frac{2}{5}$ days. How many days will it take each working alone?

57. A alone can do a piece of work in 15 hours; A and B together can do it in 9 hours, and A and C together in 10 hours. A commences work at 6 A.M.; at what hour can he be relieved by B and C, so that the work may be completed at 8 P.M.?

58. A man invests $\frac{5}{12}$ of a certain sum in $4\frac{1}{2}$ per cent bonds, and the balance in $3\frac{1}{2}$ per cent bonds, and finds his annual income to be \$ 117.50. How much does he invest in each kind of bond?

(The annual income from p dollars, invested at r per cent, is represented by $\frac{pr}{100}$.)

59. An express train whose rate is 36 miles an hour starts 54 minutes after a slow train, and overtakes it in 1 hour 48 minutes. What is the rate of the slow train?

60. At what time between 10 and 11 is the minute-hand of a watch 25 minutes in advance of the hour-hand?

61. A woman sells half an egg more than half her eggs. She then sells half an egg more than half her remaining eggs. A third time she does the same, and now she has sold all her eggs. How many had she at first?

62. A man invests two-fifths of his money in $6\frac{1}{4}$ per cent bonds, two-ninths in $5\frac{1}{4}$ per cent bonds, and the balance in $3\frac{3}{4}$ per cent bonds. His income from the investments is \$ 915. Find the amount of his property.

63. A man starts in business with $8000, and adds to his capital annually one-fourth of it. At the end of each year he sets aside a fixed sum for expenses. At the end of three years, after deducting the fixed sum for expenses, his capital is reduced to $6475. What are his annual expenses?

64. If 19 oz. of gold weigh 18 oz. in water, and 10 oz. of silver weigh 9 oz. in water, how many ounces of each metal are there in a mixed mass weighing 127 oz. in air, and 117 oz. in water?

65. A fox is pursued by a hound, and has a start of 63 of her own leaps. The fox makes 4 leaps while the hound makes 3; but the hound in 5 leaps goes as far as the fox in 9. How many leaps does each make before the hound catches the fox?

(Let $4x =$ the number of leaps made by the fox, and $3x =$ the number made by the hound.)

66. A merchant increases his capital annually by one-third of it, and at the end of each year sets aside $2700 for expenses. At the end of three years, after deducting the sum for expenses, he has $\frac{17}{64}$ of his original capital. Find his original capital.

PROBLEMS INVOLVING LITERAL EQUATIONS.

156. 1. Divide a into two parts such that m times the first shall exceed n times the second by b.

Let $x =$ one part.

Then, $a - x =$ the other part.

By the conditions, $mx = n(a - x) + b.$

$$mx = an - nx + b.$$

$$mx + nx = an + b.$$

$$x(m + n) = an + b.$$

Whence, $$x = \frac{an + b}{m + n}, \text{ one part,}$$

and $$a - x = a - \frac{an + b}{m + n} = \frac{am + an - an - b}{m + n}$$
$$= \frac{am - b}{m + n}, \text{ the other part.}$$

Note. The results can be used as *formulæ* for solving any problem of the above form.

Thus, let it be required to divide 25 into two parts such that 4 times the first shall exceed 3 times the second by 37.

Here, $a = 25$, $m = 4$, $n = 3$, and $b = 37$.

Substituting these values in the results of Ex. 1,

the first part $$= \frac{25 \times 3 + 37}{7} = \frac{75 + 37}{7} = \frac{112}{7} = 16,$$

and the second part $$= \frac{25 \times 4 - 37}{7} = \frac{100 - 37}{7} = \frac{63}{7} = 9.$$

2. Divide a into two parts such that m times the first shall equal n times the second.

3. A is m times as old as B, and a years ago he was n times as old. Find their ages at present.

4. A can do a piece of work in m hours, which B can do in n hours. In how many hours can it be done by both working together?

5. A vessel can be filled by three taps; by the first alone in a minutes, by the second alone in b minutes, and by the third alone in c minutes. In how many minutes will it be filled if all the taps be opened?

6. A has m dollars, and B has n dollars. After giving A a certain sum, B has r times as much money as A. What sum was given to A?

7. A gentleman distributing some money among beggars, found that in order to give them a cents each, he would need b cents more. He therefore gave them c cents each, and had d cents left. How many beggars were there?

8. A man has a hours at his disposal. How far can he ride in a coach which travels b miles an hour, so as to return home in time, walking back at the rate of c miles an hour?

9. A courier who travels a miles in a day is followed after n days by another who travels b miles in a day. In how many days will the second overtake the first?

10. What principal at r per cent interest will amount to a dollars in t years?

11. In how many years will p dollars amount to a dollars at r per cent interest?

12. At what rate per cent will p dollars amount to a dollars in t years?

13. Divide a into two parts, such that one divided by the other may give b as a quotient and c as a remainder.

14. Two men, A and B, a miles apart, start at the same time, and travel towards each other. A travels at the rate of m miles an hour, and B at the rate of n miles an hour. How far will each have travelled when they meet?

15. A grocer mixes a pounds of coffee worth m cents a pound, b pounds worth n cents a pound, and c pounds worth p cents a pound. Find the cost per pound of the mixture.

16. A banker has two kinds of money. It takes a pieces of the first kind to make a dollar, and b pieces of the second kind. If he is offered a dollar for c pieces, how many of each kind must he give?

17. Divide a into three parts, such that the first may be m times the second, and the second n times the third.

18. A and B together can do a piece of work in m hours, B and C in n hours, and C and A in p hours. In how many hours can each alone do the work?

XIV. SIMULTANEOUS EQUATIONS.

CONTAINING TWO UNKNOWN QUANTITIES.

157. If a rational and integral monomial (§ 69) involves two or more letters, its *degree with respect to them* is denoted by the sum of their exponents.

Thus, $2a^2b^4xy^3$ is of the *fourth* degree with respect to x and y.

158. If each term of an equation containing one or more unknown quantities is rational and integral, the *degree* of the equation is the degree of its term of highest degree.

Thus, if x and y represent unknown quantities,

$ax - by = c$ is an equation of the *first* degree.

$x^2 + 4x = -2$ is an equation of the *second* degree.

$2x^2 - 3xy^2 = 5$ is an equation of the *third* degree; for the term $3xy^2$ is the term of highest degree, and $3xy^2$ is of the third degree.

159. An equation containing two or more unknown quantities is satisfied by an indefinitely great number of sets of values of these quantities.

Consider, for example, the equation $x + y = 5$.

If $x = 1$, we have $1 + y = 5$, or $y = 4$.

If $x = 2$, we have $2 + y = 5$, or $y = 3$; and so on.

Thus the equation is satisfied by any one of the sets of values

$$x = 1, \quad y = 4;$$

$$x = 2, \quad y = 3; \text{ etc.}$$

For this reason, an equation containing two or more unknown quantities is called an *indeterminate equation.*

160. Two equations, each containing two unknown quantities, are said to be **Independent** when one of them is satisfied by sets of values of the unknown quantities which do not satisfy the other.

Consider, for example, the equations $x + y = 5$, $x - y = 3$.

The first equation is satisfied by the set of values $x = 3$, $y = 2$, which does not satisfy the second.

Therefore, the equations are independent.

But the equations $x + y = 5$, $2x + 2y = 10$, are not independent; for the second equation can be reduced to the form of the first by dividing each term by 2; and hence every set of values of x and y which satisfies one equation will also satisfy the other.

161. Let there be *two* independent equations (§ 160), each of the first degree, containing the unknown quantities x and y, as $x + y = 5$, $x - y = 3$.

By § 159, each equation considered by itself is satisfied by an indefinitely great number of sets of values of x and y.

But there is *only one* set of values of x and y, *i.e.*, $x = 4$, $y = 1$, which satisfies *both equations at the same time.*

A series of equations is called **Simultaneous** when each contains two or more unknown quantities, and every equation of the series is satisfied by the same set of values of the unknown quantities.

162. To *solve* a series of simultaneous equations is to find the set of values of the unknown quantities involved which satisfies all the equations at the same time.

163. Two independent, simultaneous equations may be solved by combining them in such a way as to form a single equation containing but *one* unknown quantity.

This operation is called **Elimination**.

There are three principal methods of elimination.

164. I. **Elimination by Addition or Subtraction.**

1. Solve the equations

$$\begin{cases}5x-3y=19. & (1)\\ 7x+4y=\ \ 2. & (2)\end{cases}$$

$$\begin{array}{lrcll}
\text{Multiplying (1) by 4,} & 20x-12y & = & 76. & (3)\\
\text{Multiplying (2) by 3,} & 21x+12y & = & 6. & (4)\\ \hline
\text{Adding (3) and (4),} & 41x & = & 82. & \\
\text{Whence,} & x & = & 2. & \\
\text{Substituting the value of } x \text{ in (1),} & 10-3y & = & 19. & \\
 & -3y & = & 9. & \\
\text{Whence,} & y & = & -3. &
\end{array}$$

The above is an example of elimination by *addition*.

2. Solve the equations

2 $$\begin{cases}15x+8y=\ \ \ 1. & (1)\\ 10x-7y=-24. & (2)\end{cases}$$

$$\begin{array}{lrcll}
\text{Multiplying (1) by 2,} & 30x+16y & = & 2. & (3)\\
\text{Multiplying (2) by 3,} & 30x-21y & = & -72. & (4)\\ \hline
\text{Subtracting (4) from (3),} & 37y & = & 74. & \\
\text{Whence,} & y & = & 2. & \\
\text{Substituting the value of } y \text{ in (2),} & 10x-14 & = & -24. & \\
 & 10x & = & -10. & \\
\text{Whence,} & x & = & -1. &
\end{array}$$

The above is an example of elimination by *subtraction*.

RULE.

If necessary, multiply the given equations by such numbers as will make the coefficients of one of the unknown quantities in the resulting equations of equal absolute value.

Add or subtract the resulting equations according as the coefficients of equal absolute value are of unlike or like sign.

Note. If the coefficients which are to be made of equal absolute value are prime to each other, each may be used as the multiplier for the other equation; but if they are not prime to each other, such multipliers should be used as will produce their lowest common multiple. Thus, in Ex. 1, to make the coefficients of y of equal absolute value, we multiplied (1) by 4 and (2) by 3; but in Ex. 2, to make the coefficients of x of equal absolute value, since the L. C. M. of 10 and 15 is 30, we multiplied (1) by 2 and (2) by 3.

EXAMPLES.

Solve by the method of addition or subtraction:

3. $\begin{cases} 5x+4y=22. \\ 3x+y=9. \end{cases}$

4. $\begin{cases} x-6y=-10. \\ 2x-7y=-15. \end{cases}$

5. $\begin{cases} 7x-2y=31. \\ 4x+3y=-3. \end{cases}$

6. $\begin{cases} 6x+11y=-28. \\ 5y-18x=8. \end{cases}$

7. $\begin{cases} 6x+2y=-3. \\ 5x-3y=-6. \end{cases}$

8. $\begin{cases} 4x+15y=7. \\ 14x+6y=9. \end{cases}$

9. $\begin{cases} 12x-5y=10. \\ 30x+11y=-69. \end{cases}$

10. $\begin{cases} 3x+7y=2. \\ 7x+8y=-2. \end{cases}$

11. $\begin{cases} 17x+10y=-30. \\ 13x-35y=-40. \end{cases}$

12. $\begin{cases} 11x-5y=4. \\ 9x+6y=10. \end{cases}$

13. $\begin{cases} 8x+9y=-4. \\ 8y-9x=77. \end{cases}$

14. $\begin{cases} 5x-9y=1. \\ 8x-10y=-5. \end{cases}$

15. $\begin{cases} 21x-8y=92. \\ 9x+17y=19. \end{cases}$

16. $\begin{cases} 10x-11y=-27. \\ 10y-11x=36. \end{cases}$

17. $\begin{cases} 22x+15y=9. \\ 18x+25y=71. \end{cases}$

18. $\begin{cases} 5x-24y=-123. \\ 19x-36y=-81. \end{cases}$

165. II. **Elimination by Substitution.**

1. Solve the equations $\begin{cases} 7x-9y=15. & (1) \\ 8y-5x=-17. & (2) \end{cases}$

Transposing $-5x$ **in (2),** $8y=5x-17.$

Whence, $y=\dfrac{5x-17}{8}.$ (3)

Substituting this in (1), $7x-9\left(\dfrac{5x-17}{8}\right)=15.$

Clearing of fractions, $56x-9(5x-17)=120.$

Expanding, $56x-45x+153=120.$

$11x=-33.$

Whence, $x=-3.$

Substituting the value of x **in (3),** $y=\dfrac{-15-17}{8}=-4.$

Rule.

From one of the given equations find the value of one of the unknown quantities in terms of the other, and substitute this value in place of that quantity in the other equation.

EXAMPLES.

Solve by the method of substitution:

2. $\begin{cases} 3x+2y=17. \\ 4x+y=16. \end{cases}$

3. $\begin{cases} x-6y=2. \\ 3y-8x=29. \end{cases}$

4. $\begin{cases} 2x-3y=-14. \\ 3x+7y=48. \end{cases}$

5. $\begin{cases} 8x+5y=5. \\ 3x-2y=29. \end{cases}$

6. $\begin{cases} 2x+5y=13. \\ 7x-4y=-19. \end{cases}$

7. $\begin{cases} 3x+7y=-23. \\ 5x+4y=-23. \end{cases}$

8. $\begin{cases} 5x+9y=8. \\ 6y-9x=-7. \end{cases}$

9. $\begin{cases} 5x+8y=-6. \\ 10x-12y=-5. \end{cases}$

10. $\begin{cases} 8x-3y=-6. \\ 4x+6y=7. \end{cases}$

11. $\begin{cases} 7x+8y=-10. \\ 11x+6y=-19. \end{cases}$

12. $\begin{cases} 6x-10y=5. \\ 15y-14x=-15. \end{cases}$

13. $\begin{cases} 9x+8y=-6. \\ 12x+10y=-7. \end{cases}$

14. $\begin{cases} 16x-11y=56. \\ 12x-7y=37. \end{cases}$

15. $\begin{cases} 7x-8y=-43. \\ 5y-6x=35. \end{cases}$

16. $\begin{cases} 6x-9y=19. \\ 15x+7y=-41. \end{cases}$

17. $\begin{cases} 5x-8y=60. \\ 6x+7y=-11. \end{cases}$

166. III. **Elimination by Comparison.**

1. Solve the equations $\begin{cases} 2x-5y=-16. & (1) \\ 3x+7y=5. & (2) \end{cases}$

Transposing $-5y$ in (1), $2x=5y-16.$

Whence, $x=\dfrac{5y-16}{2}.$ (3)

Transposing $7y$ in (2), $3x=5-7y.$

Whence, $x = \dfrac{5 - 7y}{3}$.

Equating the values of x, $\dfrac{5y - 16}{2} = \dfrac{5 - 7y}{3}$.

Clearing of fractions, $15y - 48 = 10 - 14y$.

$$29y = 58.$$

Whence, $y = 2$.

Substituting the value of y in (3), $x = \dfrac{10 - 16}{2} = -3$.

Rule.

From each of the given equations find the value of the same unknown quantity in terms of the other, and place these values equal to each other.

EXAMPLES.

Solve by the method of comparison:

2. $\begin{cases} 2x + y = 9. \\ 5x + 3y = 25. \end{cases}$

3. $\begin{cases} x + 2y = -2. \\ 4x - 7y = 37. \end{cases}$

4. $\begin{cases} 6x - 5y = -10. \\ 5x - 2y = -17. \end{cases}$

5. $\begin{cases} 11x + 4y = 3. \\ 8x + 9y = -10. \end{cases}$

6. $\begin{cases} 7x + 3y = -9. \\ 6y - 9x = 28. \end{cases}$

7. $\begin{cases} 12x - 25y = 1. \\ 4x + 10y = -7. \end{cases}$

8. $\begin{cases} 6x - 5y = 1. \\ 9x + 10y = 12. \end{cases}$

9. $\begin{cases} 3x - 8y = -17. \\ 7x + 6y = -15. \end{cases}$

10. $\begin{cases} 12x - 6y = 19. \\ 4y - 3x = -11. \end{cases}$

11. $\begin{cases} 6x - 7y = -12. \\ 10x - 9y = -12. \end{cases}$

12. $\begin{cases} 15x + 8y = -14. \\ 6x + 12y = 1. \end{cases}$

13. $\begin{cases} 5x + 3y = 27. \\ 8y - 3x = -26. \end{cases}$

14. $\begin{cases} 2x + 5y = -27. \\ 11x + 6y = -41. \end{cases}$

15. $\begin{cases} 8x - 9y = 6. \\ 7x + 4y = 29. \end{cases}$

16. $\begin{cases} 10x + 18y = -11. \\ 14y - 15x = -4. \end{cases}$

17. $\begin{cases} 9x - 7y = -85. \\ 4x - 11y = -93. \end{cases}$

MISCELLANEOUS EXAMPLES.

167. Before applying either method of elimination, each of the given equations should be reduced to its simplest form.

1. Solve the equations
$$\begin{cases} \dfrac{7}{x+3} - \dfrac{3}{y+4} = 0. & (1) \\ x(y-2) - y(x-5) = -13. & (2) \end{cases}$$

From (1), $7y + 28 - 3x - 9 = 0$, or $7y - 3x = -19$. (3)

From (2), $xy - 2x - xy + 5y = -13$, or $5y - 2x = -13$. (4)

Multiplying (3) by 2, $14y - 6x = -38$. (5)

Multiplying (4) by 3, $15y - 6x = -39$. (6)

Subtracting (5) from (6), $y = -1$.

Substituting the value of y in (4), $-5 - 2x = -13$.

$$-2x = -8.$$

Whence, $x = 4$.

Solve the following:

2. $\begin{cases} \dfrac{2x}{3} + \dfrac{3y}{4} = -\dfrac{7}{2}. \\ \dfrac{x}{4} - \dfrac{2y}{5} = \dfrac{11}{2}. \end{cases}$

3. $\begin{cases} 8x + 7y = 12. \\ \dfrac{x+2y}{4} + \dfrac{2x+y}{3} = 1. \end{cases}$

4. $\begin{cases} \dfrac{x+5y}{13} - \dfrac{2y+x}{11} = -1. \\ 3x - y = 2. \end{cases}$

5. $\begin{cases} \dfrac{3}{x-1} + \dfrac{4}{y-1} = 0. \\ \dfrac{5}{2x-3} - \dfrac{7}{2y+13} = 0. \end{cases}$

6. $\begin{cases} \dfrac{6+x-y}{1-x-y} = -\dfrac{7}{4}. \\ 2x + 3y = -1. \end{cases}$

7. $\begin{cases} \dfrac{y}{3} - \dfrac{x}{2} = 2. \\ \dfrac{3-2x}{5} - \dfrac{4+5y}{11} = 4. \end{cases}$

8. $\begin{cases} (x+1)(y+9) - (x+5)(y-7) = 112. \\ 2x + 3y + 9 = 0. \end{cases}$

9. $$\begin{cases} \dfrac{x+y}{2}-\dfrac{x-y}{3}=8. \\[2ex] \dfrac{x+y}{3}+\dfrac{x-y}{4}=11. \end{cases}$$

10. $$\begin{cases} \dfrac{x-y}{2}=\dfrac{25}{6}-\dfrac{x+y}{3}. \\[2ex] \dfrac{x+y-9}{2}-\dfrac{y-x-6}{3}=0. \end{cases}$$

11. $$\begin{cases} 10x-\dfrac{y-5}{7}=11. \\[2ex] 8y-\dfrac{x+3}{4}=-17. \end{cases}$$

12. $$\begin{cases} \dfrac{x+y-2}{x-y}=-\dfrac{1}{3}. \\[2ex] \dfrac{3x+y-3}{2y-x}=-\dfrac{1}{11}. \end{cases}$$

13. $$\begin{cases} \dfrac{x+2}{7}+8=2x-\dfrac{y-x}{4}. \\[2ex] 3x-\dfrac{2y-3x}{3}=2y-4. \end{cases}$$

14. $$\begin{cases} \dfrac{x+y}{4}-\dfrac{7x-5y}{11}+3=0. \\[2ex] \dfrac{x}{5}-\dfrac{2y}{7}=1. \end{cases}$$

15. $$\begin{cases} \dfrac{8-y}{5}-\dfrac{2x+3}{4}=\dfrac{y+3}{4}. \\[2ex] \dfrac{1+4y}{11}-\dfrac{x+7}{3}=3. \end{cases}$$

16. $$\begin{cases} \dfrac{2}{3}(3x+y)-\dfrac{1}{2}(2x+y+1)=\dfrac{1}{12}. \\[2ex] 3x-\dfrac{3}{2}\left(4x+y+\dfrac{5}{6}\right)=5y. \end{cases}$$

17. $$\begin{cases} 3x-\dfrac{1}{4}(2x+y+6)=5y. \\[2ex] \dfrac{1}{6}(3x+2y)-\dfrac{1}{4}(2y-x)=-8. \end{cases}$$

18. $$\begin{cases} x+2y+4-\dfrac{1}{4}\left[2x-3\left(y+\dfrac{1}{2}\right)\right]=0. \\[2ex] \dfrac{1}{2}\left(\dfrac{x}{2}-y\right)-\dfrac{1}{5}(x+2)=\dfrac{11}{10}. \end{cases}$$

19. $$\begin{cases} \dfrac{x-3y}{2}-\dfrac{y-3x}{2}-8=0. \\[2ex] \dfrac{\frac{1}{5}x+\frac{3}{4}y}{\frac{1}{2}x-\frac{11}{3}y}=-\dfrac{7}{30}. \end{cases}$$

20. $$\begin{cases} .8x+.05y=.6365. \\ .09x-.4y=.1. \end{cases}$$

21. $$\begin{cases} 3x-\dfrac{2x-y}{7}=5y-1-\dfrac{5x-2y}{4}. \\[2ex] \dfrac{7x+3y+12}{x-5y-4}=-3. \end{cases}$$

22. $$\begin{cases} 2-\dfrac{2x+3y}{3}=x+y-\dfrac{5x-2y}{17}. \\[2ex] \dfrac{3x+4y}{3x-4y}-\dfrac{5}{13}=0. \end{cases}$$

8 4 7

8

[illegible]

11 17

23. $$\begin{cases} \dfrac{5x+6}{10}-\dfrac{11y-5}{21}=11. \\[2ex] \dfrac{7x}{5}-\dfrac{55y-12}{25}=37. \end{cases}$$

24. $$\begin{cases} \dfrac{x-2}{5}-\dfrac{10-x}{3}-\dfrac{y-10}{4}=0. \\[2ex] \dfrac{y+2}{6}-\dfrac{2x+y}{32}-\dfrac{x+13}{16}=0. \end{cases}$$

25. $$\begin{cases} \dfrac{2xy-3}{x+2}+\dfrac{4y+5}{x-3}=2y. \\[2ex] (2x-3y+1)(3x+11y)+25y^2=(3x+8y)(2x-y) \end{cases}$$

26. $$\begin{cases} .32y-2.4x-\dfrac{.005y+2.6}{.25}=-.8x-\dfrac{.36x+.05}{.5}. \\[2ex] \dfrac{.07x+.1}{.6}+\dfrac{.04y+.1}{.3}=0. \end{cases}$$

35

168. Literal Simultaneous Equations.

In solving literal simultaneous equations, the method of elimination by addition or subtraction is usually the best.

1. Solve the equations

$$\begin{cases} ax + by = c. & (1) \\ a'x + b'y = c'. & (2) \end{cases}$$

Multiplying (1) by b', $ab'x + bb'y = b'c.$

Multiplying (2) by b, $a'bx + bb'y = bc'.$

Subtracting, $ab'x - a'bx = b'c - bc'.$

Whence, $x = \dfrac{b'c - bc'}{ab' - a'b}.$

Multiplying (1) by a', $aa'x + a'by = a'c.$ (3)

Multiplying (2) by a, $aa'x + ab'y = ac'.$ (4)

Subtracting (3) from (4), $ab'y - a'by = ac' - a'c.$

Whence, $y = \dfrac{ac' - a'c}{ab' - a'b}.$

EXAMPLES.

Solve the following:

2. $\begin{cases} 3x + 4y = 7a. \\ 2x - 5y = 6b. \end{cases}$

3. $\begin{cases} ax - by = 1. \\ bx + ay = 1. \end{cases}$

4. $\begin{cases} mx + ny = p. \\ m'x + n'y = p'. \end{cases}$

5. $\begin{cases} ax + by = m. \\ cx - dy = n. \end{cases}$

6. $\begin{cases} (a-b)x - by = a^2 - ab. \\ x + y = 2a. \end{cases}$

7. $\begin{cases} \dfrac{x + ay}{b - 2} = a. \\ \dfrac{bx + ay}{b} = -a. \end{cases}$

8. $\begin{cases} \dfrac{x}{m} - \dfrac{y}{n} = -1. \\ \dfrac{x}{3m} - \dfrac{y}{6n} = -\dfrac{2}{3}. \end{cases}$

9. $\begin{cases} \dfrac{x}{a} - \dfrac{y}{b} = \dfrac{1}{c}. \\ \dfrac{x}{a'} + \dfrac{y}{b'} = \dfrac{1}{c'}. \end{cases}$

10. $\begin{cases} (a-b)x - (a+b)y = a^2 + b^2. \\ ay + bx = 0. \end{cases}$

11. $\begin{cases} ax - by = 2\,ab. \\ 2\,bx + 2\,ay = 3\,b^2 - a^2. \end{cases}$ 12. $\begin{cases} x - ay = b(1 + ab). \\ bx + y = a(1 + ab). \end{cases}$

13. $\begin{cases} \dfrac{x}{3\,a} + \dfrac{y}{3\,b} = a + b. \\ x - y = 2(a^2 - b^2). \end{cases}$

14. $\begin{cases} (b - a)x - (a - c)y = bc - a^2. \\ (b - c)x - ay = -\,ac. \end{cases}$

15. $\begin{cases} (b + c)x + (b - c)y = 2\,ab. \\ (a + c)x - (a - c)y = 2\,ac. \end{cases}$

16. $\begin{cases} mx + ny = mn(m^2 + n^2). \\ x + y = mn(m + n). \end{cases}$

17. $\begin{cases} ax - by = 2\,b. \\ bx + ay = \dfrac{a^3 - a^2b + ab^2 + b^3}{ab}. \end{cases}$

18. $\begin{cases} (a + b)x - (a - b)y = 3\,ab. \\ (a - b)x - (a + b)y = ab. \end{cases}$

19. $\begin{cases} \dfrac{2x - b}{a} = \dfrac{3\,x - y}{a + 2\,b} \\ \dfrac{2\,x - b}{a} = \dfrac{a - 2\,y}{b}. \end{cases}$

169. Certain equations in which the unknown quantities occur in the denominators of fractions may be readily solved without previously clearing of fractions.

1. Solve the equations $\begin{cases} \dfrac{10}{x} - \dfrac{9}{y} = 8. & (1) \\ \dfrac{8}{x} + \dfrac{15}{y} = -1. & (2) \end{cases}$

Multiplying (1) by 5, $\frac{50}{x}-\frac{45}{y}=40.$

Multiplying (2) by 3, $\frac{24}{x}+\frac{45}{y}=-3.$

Adding, $\frac{74}{x}=37.$

Then, $74=37x$, and $x=2$.

Substituting the value of x in (1), $5-\frac{9}{y}=8.$

Then, $-\frac{9}{y}=3$, and $y=-3$.

EXAMPLES.

Solve the following:

2. $$\begin{cases}\frac{9}{x}+\frac{10}{y}=-1.\\ \frac{6}{x}+\frac{15}{y}=1.\end{cases}$$

3. $$\begin{cases}2x+\frac{5}{y}=-11.\\ 4x-\frac{3}{y}=\frac{21}{2}.\end{cases}$$

4. $$\begin{cases}\frac{10}{x}-\frac{9}{y}=4.\\ \frac{8}{x}-\frac{15}{y}=\frac{9}{2}.\end{cases}$$

5. $$\begin{cases}\frac{a}{x}-\frac{b}{y}=c.\\ \frac{b}{x}+\frac{a}{y}=c.\end{cases}$$

6. $$\begin{cases}\frac{m}{x}-\frac{n}{y}=p.\\ \frac{m'}{x}-\frac{n'}{y}=p'.\end{cases}$$

7. $$\begin{cases}\frac{5}{3x}-\frac{7}{y}=\frac{29}{9}.\\ \frac{3}{x}+\frac{5}{4y}=-\frac{9}{8}.\end{cases}$$

9. $$\begin{cases}\frac{a}{bx}+\frac{b}{ay}=0.\\ \frac{b}{ax}+\frac{a}{by}=\frac{b^4-a^4}{a^2b^2}.\end{cases}$$

8. $$\begin{cases}\frac{5}{2x}-\frac{4}{3y}=\frac{1}{2}.\\ \frac{2}{3x}-\frac{1}{2y}=\frac{7}{72}.\end{cases}$$

10. $$\begin{cases}\frac{a+b}{x}-\frac{1}{ay}=-\frac{b}{a}.\\ \frac{ab-b^2}{x}+\frac{1}{y}=\frac{a^2+3ab}{a+b}.\end{cases}$$

11. $$\begin{cases}\frac{a+b}{x}+\frac{a-b}{y}=5b-a.\\ \frac{a}{x}+\frac{b}{y}=2a-3b.\end{cases}$$

XV. SIMULTANEOUS EQUATIONS.

CONTAINING MORE THAN TWO UNKNOWN QUANTITIES.

170. If we have *three* independent simple equations, containing *three* unknown quantities, we may combine any two of them by one of the methods of elimination explained in §§ 164 to 166, so as to obtain a single equation containing only two unknown quantities.

We may then combine the remaining equation with either of the other two, and obtain another equation containing the same two unknown quantities.

By solving the two equations containing two unknown quantities, we may obtain their values; and substituting them in either of the given equations, the value of the remaining unknown quantity may be found.

We proceed in a similar manner when the number of equations and of unknown quantities is greater than three.

The method of elimination by addition or subtraction is usually the most convenient.

1. **Solve the equations**
$$\left\{\begin{array}{r}6x-4y-7z=17.\\9x-7y-16z=29.\\10x-5y-3z=23.\end{array}\right\}\quad\begin{array}{r}(1)\\(2)\\(3)\end{array}$$

$$\begin{array}{lrr}
\text{Multiplying (1) by 3,} & 18x-12y-21z=\ \ 51. & \\
\text{Multiplying (2) by 2,} & \underline{18x-14y-32z=\ \ 58.} & \\
\text{Subtracting,} & 2y+11z=-7. & (4)\\
\text{Multiplying (1) by 5,} & 30x-20y-35z=\ \ 85. & (5)\\
\text{Multiplying (3) by 3,} & \underline{30x-15y-9z=\ \ 69.} & (6)\\
\text{Subtracting (5) from (6),} & 5y+26z=-16. & (7)\\
\text{Multiplying (4) by 5,} & 10y+55z=-35. & \\
\text{Multiplying (7) by 2,} & \underline{10y+52z=-32.} & \\
\text{Subtracting,} & 3z=-3. &
\end{array}$$

Take the equation having [illegible] least number of terms

Whence, $z = -1.$

Substituting in (4), $2y - 11 = -7.$

Whence, $y = 2.$

Substituting in (1), $6x - 8 + 7 = 17.$

Whence, $x = 3.$

In certain cases the solution may be abridged by means of the artifice which is employed in the following example.

2. Solve the equations
$$\begin{cases} u + x + y = 6. & (1) \\ x + y + z = 7. & (2) \\ y + z + u = 8. & (3) \\ z + u + x = 9. & (4) \end{cases}$$

Adding, $3u + 3x + 3y + 3z = 30.$

Dividing by 3, $u + x + y + z = 10.$ (5)

Subtracting (2) from (5), $u = 3.$

Subtracting (3) from (5), $x = 2.$

Subtracting (4) from (5), $y = 1.$

Subtracting (1) from (5), $z = 4.$

EXAMPLES.

Solve the following:

3. $\begin{cases} 3x + 2y = 13. \\ 3y - 2z = 8. \\ 2x - 3z = 9. \end{cases}$

4. $\begin{cases} 3x + 4y + 5z = -21. \\ x + y - z = -11. \\ y - 8z = -20. \end{cases}$

5. $\begin{cases} 12x - 4y + z = 3. \\ x - y - 2z = -1. \\ 5x - 2y = 0. \end{cases}$

6. $\begin{cases} 2x - y + z = -9. \\ x - 2y + z = 0. \\ x - y + 2z = -11. \end{cases}$

7. $\begin{cases} x - y + z = 9. \\ x - 2y + 3z = 32. \\ x - 4y + 5z = 62. \end{cases}$

8. $\begin{cases} 3x - y - z = 7. \\ x - 3y - z = 21. \\ x - y - 3z = 27. \end{cases}$

9. $\begin{cases} 2x - 3y = 4. \\ 4x - 3z = 2. \\ 4y + 2z = -3. \end{cases}$

10. $\begin{cases} x + 2y - 3z = 5. \\ 3x - 22y + 6z = 4. \\ 7x - 6y - 3z = 15. \end{cases}$

11. $\begin{cases} 5x + y + 4z = -5. \\ 3x - 5y + 6z = -20. \\ x - 3y - 4z = -21. \end{cases}$

12. $\begin{cases} 2x - 3y - 4z = -10. \\ 3x + 4y + 2z = -5. \\ 4x + 2y + 3z = -21. \end{cases}$

13. $\begin{cases} 5x + 4y + 3z = 7. \\ 9x - y + 6z = -39. \\ 8x - 7y - 12z = -2. \end{cases}$

14. $\begin{cases} 2x - 6y - 5z = -11. \\ 10x + 9y - 3z = 50. \\ 4x - 8y + z = 15. \end{cases}$

15. $\begin{cases} x + 3y - 7z = 31. \\ 3x + y + 5z = -49. \\ 20x + 2y - 5z = -35. \end{cases}$

16. $\begin{cases} 9x + 4y = 10z + 11. \\ 12y - 5z = 6x - 9. \\ 15z + 3x = -8y - 16. \end{cases}$

17. $\begin{cases} 5x + 16y + 6z = 4. \\ 10x + 4y - 12z = -7. \\ 15x - 12y - 3z = -10. \end{cases}$

18. $\begin{cases} \dfrac{6}{y} + \dfrac{7}{x} = 5. \\ \dfrac{3}{2z} + \dfrac{3}{y} = 1. \\ \dfrac{7}{x} + \dfrac{27}{z} = 8. \end{cases}$

19. $\begin{cases} \dfrac{1}{x} + \dfrac{3}{2y} = \dfrac{9}{5}. \\ \dfrac{1}{y} + \dfrac{4}{3z} = \dfrac{5}{3}. \\ \dfrac{1}{z} + \dfrac{5}{4x} = \dfrac{7}{4}. \end{cases}$

20. $\begin{cases} x - ay = a(a^2 + 1). \\ y - az = a^2(a^2 - 1). \\ z - ax = -a^2(a + 1). \end{cases}$

21. $\begin{cases} a(x - c) + b(y - c) = 0. \\ b(y - a) + c(z - a) = 0. \\ c(z - b) + a(x - b) = 0. \end{cases}$

22. $\begin{cases} u + x + y = 7. \\ x + y + z = -8. \\ y + z + u = 5. \\ z + u + x = -10 \end{cases}$

23. $\begin{cases} \dfrac{1}{x} - \dfrac{1}{y} - \dfrac{1}{z} = a. \\ \dfrac{1}{y} - \dfrac{1}{z} - \dfrac{1}{x} = b. \\ \dfrac{1}{z} - \dfrac{1}{x} - \dfrac{1}{y} = c. \end{cases}$

24. $$\begin{cases} u-2x=-13. \\ x-3y=13. \\ y-4z=5. \\ z-5u=23. \end{cases}$$

25. $$\begin{cases} 7x+4y-3u=0. \\ 5x+4y+4z-5u=0. \\ 2x+z-u=0. \\ 2x+4y-3z-u=-8. \end{cases}$$

26. $$\begin{cases} x-\frac{y}{2}-\frac{z}{3}=\frac{25}{3}. \\ y-\frac{z}{2}-\frac{x}{3}=-2. \\ z-\frac{x}{2}-\frac{y}{3}=-\frac{19}{3}. \end{cases}$$

27. $$\begin{cases} 9x-26y-16z=-44. \\ 12x-8y+15z=-15. \\ 8x-9y+13z=-24. \end{cases}$$

28. $$\begin{cases} x+\frac{y}{2}-\frac{z}{3}=17. \\ x+\frac{y}{7}=6-\frac{z+2}{5}. \\ \frac{x}{3}-y-\frac{z-13}{2}=\frac{1}{2}. \end{cases}$$

29. $$\begin{cases} \frac{x}{2}+\frac{y}{3}-\frac{z}{4}=-23. \\ \frac{x}{3}-\frac{y}{4}+\frac{z}{2}=20. \\ \frac{x}{4}-\frac{y}{2}-\frac{z}{3}=-3. \end{cases}$$

30. $$\begin{cases} \frac{1}{x}+\frac{1}{y}=\frac{1}{c}. \\ \frac{1}{y}+\frac{1}{z}=\frac{1}{a}. \\ \frac{1}{z}+\frac{1}{x}=\frac{1}{b}. \end{cases}$$

31. $$\begin{cases} \frac{x}{a}+\frac{y}{c}=2b. \\ \frac{x}{b}+\frac{z}{c}=2a. \\ \frac{y}{b}+\frac{z}{a}=2c. \end{cases}$$

32. $$\begin{cases} \frac{b}{x}+\frac{a}{y}=1. \\ \frac{a}{z}+\frac{c}{x}=1. \\ \frac{c}{y}+\frac{b}{z}=1. \end{cases}$$

33. $$\begin{cases} x+y+az=a+2. \\ ay+az+a^2x=a^3+a+1. \\ az+ax+a^2y=2a^2+1. \end{cases}$$

34. $$\begin{cases} \frac{3x-y}{5}+\frac{5y+4z}{2}=\frac{19}{2}. \\ \frac{2x+3z}{6}-\frac{x-4y}{4}=\frac{7}{4}. \\ \frac{4x-z}{3}-\frac{3y-5z}{2}=\frac{49}{3}. \end{cases}$$

XVI. PROBLEMS.

INVOLVING SIMULTANEOUS EQUATIONS.

171. In solving problems where two or more letters are used to represent unknown quantities, we must obtain from the conditions of the problem *as many independent equations* (§ 160) *as there are unknown quantities to be determined.*

172. 1. Divide 81 into two parts such that three-fifths of the greater shall exceed five-ninths of the less by 7.

Let $x =$ the greater part,

and $y =$ the less.

Since the sum of the greater and less parts is 81, we have

$$x + y = 81. \qquad (1)$$

And since three-fifths of the greater exceeds five-ninths of the less by 7,

$$\frac{3x}{5} = \frac{5y}{9} + 7. \qquad (2)$$

Solving (1) and (2), $x = 45$, $y = 36$.

2. If 3 be added to both numerator and denominator of a fraction, its value is $\frac{2}{3}$; and if 2 be subtracted from both numerator and denominator, its value is $\frac{1}{2}$. Required the fraction.

Let $x =$ the numerator,

and $y =$ the denominator.

By the conditions, $$\frac{x+3}{y+3} = \frac{2}{3},$$

and $$\frac{x-2}{y-2} = \frac{1}{2}.$$

Solving these equations, $x = 7$, $y = 12$.

Therefore, the fraction is $\frac{7}{12}$.

PROBLEMS.

3. Divide 59 into two parts such that two-thirds of the less shall be less by 4 than four-sevenths of the greater.

4. Find two numbers such that two-fifths of the greater exceeds one-half of the less by 2, and four-thirds of the less exceeds three-fourths of the greater by 1.

5. If 5 be added to the numerator of a certain fraction, its value is $\frac{5}{3}$; and if 5 be subtracted from its denominator, its value is $\frac{5}{2}$. Find the fraction.

6. If 9 be added to both terms of a fraction, its value is $\frac{6}{7}$; and if 7 be subtracted from both terms, its value is $\frac{2}{3}$. Find the fraction.

7. A grocer can sell for $57 either 9 barrels of apples and 16 barrels of flour, or 15 barrels of apples and 14 barrels of flour. Find the price per barrel of the apples and of the flour.

8. A's age is $\frac{3}{5}$ of B's; but in 16 years his age will be $\frac{5}{7}$ of B's. Find their ages at present.

9. If twice the greater of two numbers be divided by the less, the quotient is 3 and the remainder 7; and if five times the less be divided by the greater, the quotient is 2 and the remainder 23. Find the numbers.

10. If the numerator of a fraction be trebled, and the denominator increased by 8, the value of the fraction is $\frac{9}{8}$; and if the denominator be halved, and the numerator decreased by 7, the value of the fraction is $\frac{1}{4}$. Find the fraction.

11. Three years ago A's age was $\frac{4}{3}$ of B's; but in nine years his age will be $\frac{11}{9}$ of B's. Find their ages at present.

12. A and B can do a piece of work in 9 hours. After working together 7 hours, B finishes the work in 5 hours. In how many hours could each alone do the work?

Corrected

13. A man invests a certain sum of money in $4\frac{1}{2}$ per cent stock, and a sum $180 greater than the first in $3\frac{1}{2}$ per cent stock. The incomes from the two investments are equal. Find the sums invested.

14. My income and assessed taxes together amount to $64. If the income tax were increased one-fourth, and the assessed tax decreased one-fifth, they would together amount to $63.80. Find the amount of each tax.

15. If B gives A $12, A will have $\frac{2}{5}$ as much money as B; but if A gives B $12, B will have $\frac{4}{3}$ as much money as A. How much money has each?

16. A man pays with a $5 note two bills, one of which is six-sevenths of the other, and receives back in change seven times the difference of the bills. Find their amounts.

17. Find three numbers such that the first with one-third the others, the second with one-fourth the others, and the third with one-fifth the others may each be equal to 25.

18. A sum of money was divided equally among a certain number of persons. Had there been 3 more, each would have received $1 less; had there been 6 fewer, each would have received $5 more. How many persons were there, and how much did each receive?

Let $x =$ the number of persons,

and $y =$ the number of dollars received by each.

Then, $xy =$ the number of dollars divided.

The sum of money could be divided among $x + 8$ persons, each of whom would receive $y - 1$ dollars; and among $x - 6$ persons, each of whom would receive $y + 5$ dollars.

Whence, $(x + 3)(y - 1)$ and $(x - 6)(y + 5)$ will also represent the number of dollars divided.

Then $(x + 3)(y - 1) = xy,$

and $(x - 6)(y + 5) = xy.$

Solving these equations,

$$x = 12, \ y = 5.$$

19. A man bought a certain number of eggs. If he had bought 56 more for the same money, they would have cost a cent apiece less; if 24 less, a cent apiece more. How many eggs did he buy, and at what price each?

20. A boy spent his money for oranges. If he had got 15 more for his money, they would have cost $1\frac{1}{2}$ cents each less; if 5 fewer, they would have cost $1\frac{1}{2}$ cents each more. How much did he spend, and how many oranges did he get?

21. A sum of money is divided equally among a certain number of persons. Had there been m more, each would have received a dollars less; if n less, each would have received b dollars more. How many persons were there, and how much did each receive?

22. A purse contained $6.55 in quarter-dollars and dimes; after 6 quarters and 8 dimes had been taken out, there remained 3 times as many quarters as dimes. How many were there of each at first?

23. A dealer has two kinds of wine, worth 50 and 90 cents a gallon, respectively. How many gallons of each must be taken to make a mixture of 70 gallons, worth 75 cents a gallons?

24. A grocer bought a certain number of eggs at the rate of 22 cents a dozen, and seven-fifths as many at the rate of 14 cents a dozen. He sold them at the rate of 20 cents a dozen, and gained 24 cents by the transaction. How many of each kind did he buy?

25. A and B can do a piece of work in 10 days, A and C in 12 days, and B and C in 20 days. In how many days can each of them alone do it?

26. A resolution was adopted by a majority of 10 votes; but if one-fourth of those who voted for it had voted against it, it would have been defeated by a majority of 6 votes. How many voted for, and how many against it?

27. The sum of the three digits of a number is 13. If the number, decreased by 8, be divided by the sum of its second and third digits, the quotient is 25; and if 99 be added to the number, the digits will be inverted. Find the number.

Let $x =$ the first digit,

$y =$ the second,

and $z =$ the third.

Then, $100x + 10y + z =$ the number,

and $100z + 10y + x =$ the number with its digits inverted.

By the conditions of the problem,

$$x + y + z = 13,$$

$$\frac{100x + 10y + z - 8}{y + z} = 25,$$

and $$100x + 10y + z + 99 = 100z + 10y + x.$$

Solving these equations, $x = 2$, $y = 8$, $z = 3$.

Therefore, the number is 283.

28. The sum of the two digits of a number is 16; and if 18 be subtracted from the number, the digits will be inverted. Find the number.

29. The sum of the three digits of a number is 23; and the digit in the tens' place exceeds that in the units' place by 3. If 198 be subtracted from the number, the digits will be inverted. Find the number.

30. If the digits of a number of two figures be inverted, the sum of the resulting number and twice the given number is 204; and if the number be divided by the sum of its digits, the quotient is 7 and the remainder 6. Find the number.

31. If a certain number be divided by the sum of its two digits, the quotient is 4 and the remainder 3. If the digits be inverted, the quotient of the resulting number increased by 23, divided by the given number, is 2. Find the number.

32. Two vessels contain mixtures of wine and water. In one there is three times as much wine as water, and in the other five times as much water as wine. How many gallons must be taken from each to fill a third vessel whose capacity is 7 gallons, so that its contents may be half wine and half water?

33. If a lot of land were 6 feet longer and 5 feet wider, it would contain 839 square feet more; and if it were 4 feet longer and 7 feet wider, it would contain 879 square feet more. Find its length and width.

34. A and B are building a piece of fence 189 feet long. After they have worked together 9 hours A leaves off, and B finishes the work in $12\frac{3}{5}$ hours. If 12 hours had occurred before A left off, B would have finished the work in $4\frac{1}{5}$ hours. How many feet does each build in one hour?

35. The sum of the three digits of a number is 17. The sum of 3 times the first digit, 5 times the second, and 4 times the third is 70; and if 297 be added to the number, the digits will be inverted. Find the number.

36. The rate of an express train is five-thirds that of a slow train, and it travels 36 miles in 32 minutes less time than the slow train. Find the rate of each in miles an hour.

37. Divide $396 among A, B, C, and D, so that A may receive one-half the sum of the shares of B and C, B one-third the sum of the shares of C and D, and C one-fourth the sum of the shares of A and D.

38. A merchant has two casks of wine, containing together 56 gallons. He pours from the first into the second as much as the second contained at first; he then pours from the second into the first as much as was left in the first; and again from the first into the second as much as was left in the second. There is now three-fourths as much in the first as in the second. How many gallons did each contain at first?

39. A crew can row 10 miles in 50 minutes down stream, and 12 miles in an hour and a half against the stream. Find the rate in miles an hour of the current, and of the crew in still water.

Let $x =$ the number of miles an hour rowed by the crew in still water,

and $y =$ the number of miles an hour of the current.

Then, $x + y =$ the number of miles an hour of the crew rowing down stream,

and $x - y =$ the number of miles an hour of the crew rowing up stream.

Since the number of miles an hour rowed by the crew is equal to the distance divided by the time in hours, we have

$$x + y = 10 \div \frac{5}{6} = 12,$$

and

$$x - y = 12 \div \frac{3}{2} = 8.$$

Solving these equations, $x = 10,\ y = 2.$

40. A crew can row a miles in m hours down stream, and b miles in n hours against the stream. Find the rate in miles an hour of the current, and of the crew in still water.

41. A vessel can go 63 miles down stream and back again in 20 hours; and it can go 3 miles against the current in the same time that it goes 7 miles with it. Find its rate in miles an hour in going, and in returning.

42. If a number of two figures, diminished by 3, be divided by the sum of its digits, the quotient is 5. If the digits be inverted, the quotient of the resulting number increased by 18, divided by the sum of the digits, is 7. Find the number.

43. The digits of a number of three figures have equal differences in their order. If the number be divided by one-half the sum of its digits, the quotient is 41; and if 594 be added to the number, the digits will be inverted. Find the number.

44. If I were to make my field 5 feet longer and 7 feet wider, its area would be increased by 830 square feet; but if I were to make its length 8 feet less, and its width 4 feet less, its area would be diminished by 700 square feet. Find its length and width.

45. A certain sum of money at simple interest amounts in 3 years to $ 420, and in 7 years to $ 480. Required the sum and the rate of interest.

46. A certain sum of money at simple interest amounts in m years to a dollars, and in n years to b dollars. Required the sum and the rate of interest.

47. A and B together can do a piece of work in $8\frac{3}{4}$ days; but if A had worked $\frac{5}{6}$ as fast, and B $\frac{3}{2}$ as fast, they would have done it in $7\frac{7}{8}$ days. In how many days could each alone do the work?

48. A sum of money at simple interest amounts to $ 2080 in 8 months, and to $ 2150 in 15 months. Find the sum and the rate of interest.

49. A train running from A to B meets with an accident which causes its speed to be reduced to one-third of what it was before, and it is in consequence 5 hours late. If the accident had happened 60 miles nearer B, the train would have been only 1 hour late. Find the rate of the train before the accident, and the distance to B from the point of detention.

Let $3x =$ the number of miles an hour of the train before the accident.

Then, $x =$ the number of miles an hour after the accident.

Let $y =$ the number of miles to B from the point of detention.

The train would have done the last y miles of its journey in $\frac{y}{3x}$ hours; but owing to the accident, it does the distance in $\frac{y}{x}$ hours.

Then,
$$\frac{y}{x} = \frac{y}{3x} + 5. \qquad (1)$$

If the accident had occurred 60 miles nearer B, the distance to B from the point of detention would have been $y - 60$ miles.

Had there been no accident, the train would have done this in $\frac{y-60}{3x}$ hours, and the accident would have increased the time to $\frac{y-60}{x}$ hours.

Then, $$\frac{y-60}{x} = \frac{y-60}{3x} + 1. \qquad (2)$$

Subtracting (2) from (1), $\frac{60}{x} = \frac{60}{3x} + 4$, or $\frac{40}{x} = 4$.

Whence, $x = 10$.

Then the rate of the train before the accident was 30 miles an hour.

Substituting in (1), $\frac{y}{10} = \frac{y}{30} + 5$, or $\frac{y}{15} = 5$.

Whence, $y = 75$.

50. A train running from A to B meets with an accident which delays it 45 minutes; it then proceeds at five-sixths its former rate, and arrives at B 75 minutes late. Had the accident occurred 45 miles nearer A, the train would have been 90 minutes late. Find the rate of the train before the accident, and the distance to B from the point of detention.

51. The unit's digit of a number of three digits is 7. If the digits in the hundreds' and tens' places be interchanged, the number is decreased by 180. If the digit in the hundreds' place be halved, and the other two digits interchanged, the number is decreased by 273. Find the number.

52. A, B, C, and D play at cards, having together $46. After A has won one-third of B's money, B one-fourth of C's, and C one-fifth of D's, A, B, and C have each $10. How much had each at first?

53. A, B, and C have together $24. A gives to B and C as much as each of them has; B gives to A and C as much as each of them then has; and C gives to A and B as much as each of them then has. They have now equal amounts. How much did each have at first?

54. The fore-wheel of a carriage makes 8 revolutions more than the hind-wheel in going 180 feet; but if the circumference of the fore-wheel were $\frac{4}{3}$ as great, and of the hind-wheel $\frac{6}{5}$ as great, the fore-wheel would make only 5 revolutions more than the hind-wheel in going the same distance. Find the circumference of each wheel.

55. A and B together can do a piece of work in m days, B and C in n days, and C and A in p days. In what time can each alone perform the work?

56. A piece of work can be completed by A working 3 days, B 7 days, and C 1 day; by A working 5 days, B 1 day, and C 7 days; or by A working 1 day, B 5 days, and C 11 days. In how many days can each alone perform the work?

57. A man has a sum of money invested at a certain rate of interest. Another man has a sum greater by $ 3000, invested at a rate 1 per cent less, and his income is $ 45 less than that of the first. A third man has a sum less by $ 2000 than that of the first, invested at a rate 1 per cent greater, and his income is $ 40 greater than that of the first. Find the capital of each man, and the rate at which it is invested.

58. A and B can do a piece of work in a hours. After working together b hours, B finishes the work in c hours. In how many hours could each alone do the work?

59. A crew row up stream 26 miles and down stream 35 miles in 9 hours. They then row up stream 32 miles and down stream 28 miles in 10 hours. Find the rate in miles an hour of the current, and of the crew in still water.

(Let x and y represent the number of miles an hour of the crew rowing up and down stream, respectively.)

60. A sum of money, at 6 per cent interest, amounts to $ 5900 for a certain time, and to $ 7100 for a time longer by 4 years. Find the principal and the time.

61. A gives to B and C twice as much money as each of them has; B gives to A and C twice as much as each of them then has; and C gives to A and B twice as much as each of them then has. Each has now $27. How much did each have at first?

62. A party at a tavern found, on paying their bill, that had there been 4 more, each would have paid 75 cents less; but if there had been 4 fewer, each would have paid $1.50 more. How many were there, and how much did each pay?

63. An express train travels 30 miles in 27 minutes less time than a slow train. If the rate of the express train were $\frac{5}{4}$ as great, and of the slow train $\frac{4}{5}$ as great, the express train would travel 30 miles in 54 minutes less time than the slow train. Find the rate of each in miles an hour.

64. A and B run a race of 450 feet. The first heat, A gives B a start of 135 feet, and is beaten by 4 seconds; the second heat, A gives B a start of 30 feet, and beats him by 3 seconds. How many feet can each run in a second?

65. A sum of money consists of quarter-dollars, dimes, and half-dimes. Its value is as many dimes as there are pieces of money; its value is also as many quarters as there are dimes; and the number of half-dimes is one more than the number of dimes. Find the number of each coin.

66. A man invests $5100, partly in $3\frac{1}{2}$ per cent stock at $90 a share, and partly in 4 per cent stock at $120 a share, the par value of each share being $100. If his annual income is $185, how many shares of each stock does he buy?

67. A and B run a race of 336 yards. The first heat, A gives B a start of 28 yards, and beats him by 2 seconds; the second heat, A gives B a start of 12 seconds, and is beaten by 48 yards. How many yards can each run in a second?

XVII. INEQUALITIES.

173. Definitions.

The **Signs of Inequality**, $>$ and $<$, are read "*is greater than*" and "*is less than*," respectively.

Thus, $a > b$ is read "a is greater than b"; $a < b$ is read "a is less than b."

The **Sign of Continuation**, $\cdots$, signifies "*and so on*," or "*continued by the same law*."

174. One number is said to be *greater* than another when the remainder obtained by subtracting the second from the first is a *positive* number; and one number is said to be *less* than another when the remainder obtained by subtracting the second from the first is a *negative* number.

Thus, if $a - b$ is a positive number, $a > b$; and if $a - b$ is a negative number, $a < b$.

175. An **Inequality** is a statement that one of two expressions is greater or less than another.

The *First Member* of an inequality is the expression to the left of the sign of inequality; the *Second Member* is the expression to the right of that sign.

Any term of either member of an inequality is called a *term* of the inequality.

176. Two or more inequalities are said to *subsist in the same sense* when the first member is the greater or the less in both.

Thus, $a > b$ and $c > d$ subsist in the same sense.

177. *An inequality will continue in the same sense after the same quantity has been added to, or subtracted from, both members.*

For consider the inequality $a > b$.

Then by § 174, $a - b$ is a positive number.

Hence, each of the numbers

$$(a + c) - (b + c), \text{ and } (a - c) - (b - c)$$

is positive, since each is equal to $a - b$.

Therefore, $a + c > b + c$, and $a - c > b - c$. (§ 174)

178. It follows from § 177 that *a term may be transposed from one member of an inequality to the other by changing its sign.*

179. *If the signs of all the terms of an inequality be changed, the sign of inequality must be reversed.*

For consider the inequality $a - b > c - d$.

Transposing every term, we have

$$d - c > b - a. \quad (§ 178)$$

That is, $b - a < d - c$.

180. *An inequality will continue in the same sense after both members have been multiplied or divided by the same positive number.*

For consider the inequality $a > b$.

By § 174, $a - b$ is a positive number.

Hence, if m is a positive number, each of the numbers

$$m(a - b) \text{ and } \frac{a - b}{m},$$

or, $ma - mb$ and $\frac{a}{m} - \frac{b}{m}$, is positive

Therefore, $ma > mb$, and $\frac{a}{m} > \frac{b}{m}$.

181. It follows from §§ 179 and 180 that *if both members of an inequality be multiplied or divided by the same negative number, the sign of inequality must be reversed.*

182. *If any number of inequalities, subsisting in the same sense, be added member to member, the resulting inequality will also subsist in the same sense.*

For consider the inequalities $a > b$, $a' > b'$, $a'' > b''$, $\cdots$.

Then each of the numbers $a - b$, $a' - b'$, $a'' - b''$, $\cdots$, is positive.

Therefore, their sum

$$a - b + a' - b' + a'' - b'' + \cdots,$$

or, $$a + a' + a'' + \cdots - (b + b' + b'' + \cdots),$$

is a positive number.

Whence, $a + a' + a'' + \cdots > b + b' + b'' + \cdots$.

183. It is to be observed that, if two inequalities, subsisting in the same sense, be *subtracted* member from member, the resulting inequality does not necessarily subsist in the same sense.

Thus, if $a > b$ and $a' > b'$, the numbers $a-b$ and $a'-b'$ are positive.

But $(a - b) - (a' - b')$, or its equal $(a-a') - (b - b')$, may be positive, negative, or zero; and hence $a-a'$ may be greater than, less than, or equal to $b - b'$.

EXAMPLES.

184. 1. Find the limit of x in the inequality

$$7x - \frac{23}{3} < \frac{2x}{3} + 5.$$

Multiplying both members by 3 (§ 180), we have

$$21x - 23 < 2x + 15.$$

Transposing (§ 178), and uniting terms,

$$19x < 38.$$

Dividing both members by 19 (§ 180),

$$x < 2, \quad \textit{Ans.}$$

2. Find the limits of x and y in the following:

$$\begin{cases} 3x + 2y > 37. & (1) \\ 2x + 3y = 33. & (2) \end{cases}$$

Multiplying (1) by 3, $9x + 6y > 111.$

Multiplying (2) by 2, $\underline{4x + 6y = \ 66.}$

Subtracting (§ 177), $5x > 45.$

Whence, $x > 9.$

Multiplying (1) by 2, $6x + 4y > \ 74.$

Multiplying (2) by 3, $\underline{6x + 9y = \ 99.}$

Subtracting, $-5y > -25.$

Dividing both members by -5 (§ 181), $y < 5.$

Find the limits of x in the following:

3. $(6x - 1)^2 - 28 < (4x - 3)(9x + 2).$

4. $(3x + 2)(4x - 5) > (2x - 3)(6x + 1) + 5.$

5. $(5x + 1)^2 + 15 > (3x - 2)^2 + (4x + 3)^2.$

6. $(x - 2)(x - 3)(x + 4) < (x + 1)(x + 2)(x - 4).$

7. $6mx - 5an > 15am - 2nx$, if $3m + n$ is negative.

8. $\frac{x + b}{a} - \frac{x - a}{b} < 2$, if a and b are positive, and $a > b$.

Find the limits of x and y in the following:

9. $\begin{cases} 4x + 9y < 40. \\ 6x - y = 2. \end{cases}$

10. $\begin{cases} 7x + 2y > 25. \\ 3x + 5y = 19. \end{cases}$

11. Find the limits of x when

$$5x + 7 < 9x - 13, \text{ and } 11x - 20 < 6x + 25.$$

12. A certain positive integer, plus 21, is greater than 8 times the number, minus 35; and twice the number, plus 11, is less than 7 times the number, minus 19. Find the number.

13. A teacher has a number of his pupils such that 8 times their number, minus 31, is less than 3 times their number, plus 69; and 13 times their number, minus 45, is greater than 7 times their number, plus 57. How many pupils has he?

14. A shepherd has a number of sheep such that 4 times the number, minus 7, is greater than 6 times the number, minus 89; and 5 times the number, plus 3, is greater than twice the number, plus 114. How many sheep has he?

15. Prove that if a and b are unequal positive numbers,

$$\frac{a}{b}+\frac{b}{a}>2.$$

Since the square of any number is positive,

$$(a-b)^2>0.$$

That is, $a^2-2ab+b^2>0.$

Transposing $-2ab$, $a^2+b^2>2ab.$

Dividing each term of the inequality by ab (§ 180), we have

$$\frac{a}{b}+\frac{b}{a}>2.$$

16. Prove that for any value of x, except 1, $x^2+1>2x$.

17. Prove that for any value of x, except $\frac{2}{3}$, $9x^2+4>12x$.

In each of the following examples, the letters are understood as representing positive numbers.

18. Prove that $\frac{a}{2b}+\frac{2b}{a}>2$, if b is not equal to $\frac{1}{2}a$.

19. Prove that $(a+2b)(a-2b)>b(6a-13b)$, if b is not equal to $\frac{1}{3}a$.

20. Prove that $a(9a-4b)>4b(2a-b)$, if b is not equal to $\frac{3}{2}a$.

21. Prove that $(a^2-b^2)(c^2-d^2)<(ac-bd)^2$, if bc does not equal ad.

XVIII. INVOLUTION.

185. Involution is the process of raising a given expression to any required power whose exponent is a positive integer.

This may be effected, as is evident from § 6, by taking the expression as a factor as many times as there are units in the exponent of the required power.

INVOLUTION OF MONOMIALS.

186. 1. Find the value of $(5a^3b^2)^3$.

By § 6, $(5a^3b^2)^3 = 5a^3b^2 \times 5a^3b^2 \times 5a^3b^2 = 125a^9b^6$, *Ans.*

2. Find the value of $(-a)^4$.

$(-a)^4 = (-a) \times (-a) \times (-a) \times (-a) = a^4$ (§ 49), *Ans.*

3. Find the value of $(-3m^4)^3$.

$(-3m^4)^3 = (-3m^4) \times (-3m^4) \times (-3m^4) = -27m^{12}$ (§ 49), *Ans.*

From the above examples, we derive the following rule:

Raise the absolute value of the numerical coefficient to the required power, and multiply the exponent of each letter by the exponent of the required power.

Give to every power of a positive term, and to every EVEN *power of a negative term, the positive sign, and to every* ODD *power of a negative term the negative sign.*

EXAMPLES.

Find the values of the following:

4. $(a^3b^4c^2)^3$.
5. $(x^5y^6z^7)^{10}$.
6. $(-m^8np^9)^7$.
7. $(-12a^{2m}b^{3n})^2$.
8. $(2m^2n^7)^5$.
9. $(-a^2b^3c^4)^9$.
10. $(x^myz^2)^n$.
11. $(-3x^2y^4z^3)^6$.
12. $(pq^{5m}r^{4n})^{11}$.
13. $(-6x^9y^8z^7)^3$.
14. $(2a^rx^{2s})^8$.
15. $(-5mn^3p^5)^4$.

A fraction may be raised to any required power by *raising both numerator and denominator to the required power, and dividing the first result by the second.*

129

16. Find the value of $\left(-\frac{2x^m}{3y^2}\right)^4$.

We have, $\left(-\frac{2x^m}{3y^2}\right)^4 = \frac{(2x^m)^4}{(3y^2)^4} = \frac{16x^{4m}}{81y^8}$, *Ans.*

Find the values of the following:

17. $\left(\frac{11a^5x^8}{9b^7y^6}\right)^2$. **19.** $\left(-\frac{2a^5b^6}{xy^8}\right)^6$. **21.** $\left(-\frac{x^5y^4}{2z^n}\right)^7$.

18. $\left(-\frac{7m^5n}{8a^6}\right)^3$. **20.** $\left(\frac{4x^6y^8}{m^4n^7}\right)^4$. **22.** $\left(\frac{3x^4yz^5}{4a^8b^6}\right)^5$.

SQUARE OF A POLYNOMIAL.

3

187. We find by multiplication:

$$\begin{array}{llllll}
a & +b & +c & & & \\
a & +b & +c & & & \\
\hline
a^2 & +ab & +ac & & & \\
 & +ab & & +b^2 & +bc & \\
 & & +ac & & +bc & +c^2 \\
\hline
a^2 & +2ab & +2ac & +b^2 & +2bc & +c^2
\end{array}$$

The result, for convenience of enunciation, may be written:

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

In like manner we find:

$$(a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd;$$

and so on.

We then have the following rule:

The square of a polynomial is equal to the sum of the squares of its terms, plus twice the product of each term by each of the following terms.

EXAMPLES.

1. Square $2x^2 - 3x - 5$.

The squares of the terms are $4x^4$, $9x^2$, and 25.

Twice the first term into each of the following terms gives the results $-12x^3$ and $-20x^2$.

Twice the second term into the following term gives the result $30x$.

$$\begin{aligned}\text{Then,}\quad (2x^2 - 3x - 5)^2 &= 4x^4 + 9x^2 + 25 - 12x^3 - 20x^2 + 30x\\ &= 4x^4 - 12x^3 - 11x^2 + 30x + 25,\ \textit{Ans.}\end{aligned}$$

Square each of the following:

2. $a - b - c$.
3. $x - y + z$.
4. $x^2 + 2x + 1$.
5. $m + 2n - 3p$.
6. $2a^2 - a + 4$.
7. $5x^2 - 3x - 1$.
8. $3x^2 + 4x + 2$.
9. $6n^3 + n - 5$.
10. $2a - 5b - c$.
11. $a^2 - 4ab + 3b^2$.
12. $2x^2 + 3xy + y^2$.
13. $x^3 + 6x^2 - 7$.
14. $4a^4 - 5a^2x^3 - 3x^6$.
15. $a - b - c - d$.
16. $a + b - c + d$.
17. $x^3 - x^2 - x + 2$.
18. $a^3 + 2a^2 - 3a - 4$.
19. $2x^3 - 5x^2 + 4x - 3$.

CUBE OF A BINOMIAL.

188. We find by multiplication:

$$\begin{array}{rl}(a+b)^2 = & a^2 + 2ab + b^2\\ & \underline{a \quad + b\qquad\qquad\quad}\\ & a^3 + 2a^2b + ab^2\\ & \underline{\qquad\quad\ \ a^2b + 2ab^2 + b^3}\\ (a+b)^3 = & a^3 + 3a^2b + 3ab^2 + b^3\end{array}$$

$$\begin{array}{rl}(a-b)^2 = & a^2 - 2ab + b^2\\ & \underline{a \quad - b\qquad\qquad\quad}\\ & a^3 - 2a^2b + ab^2\\ & \underline{\qquad\ \ - a^2b + 2ab^2 - b^3}\\ (a-b)^3 = & a^3 - 3a^2b + 3ab^2 - b^3\end{array}$$

That is, *the cube of the sum of two quantities is equal to the cube of the first, plus three times the square of the first times the second, plus three times the first times the square of the second, plus the cube of the second.*

The cube of the difference of two quantities is equal to the cube of the first, minus three times the square of the first times the second, plus three times the first times the square of the second, minus the cube of the second.

EXAMPLES.

1. Find the cube of $a + 2b$.

We have, $(a + 2b)^3 = a^3 + 3a^2(2b) + 3a(2b)^2 + (2b)^3$

$= a^3 + 6a^2b + 12ab^2 + 8b^3$, *Ans.*

2. Find the cube of $2x^3 - 3y^2$.

$$(2x^3 - 3y^2)^3 = (2x^3)^3 - 3(2x^3)^2(3y^2) + 3(2x^3)(3y^2)^2 - (3y^2)^3$$
$$= 8x^9 - 36x^6y^2 + 54x^3y^4 - 27y^6, \textit{ Ans.}$$

Cube each of the following:

3. $ax + by$.
4. $x + 2$.
5. $3a - 1$.
6. $m - 4n$.
7. $x^2 + 5$.
8. $6a - b$.
9. $5x + 2y$.
10. $4m - 3n^3$.
11. $3x^2 - 5x$.
12. $4x^4 + 5yz^3$.
13. $2x - 7x^3$.
14. $5a^6 + 6b^5$.

The cube of a *trinomial* may be found by the above method, if two of its terms be enclosed in a parenthesis and regarded as a single term.

15. Find the cube of $x^2 - 2x - 1$.

$$(x^2 - 2x - 1)^3 = [(x^2 - 2x) - 1]^3$$
$$= (x^2 - 2x)^3 - 3(x^2 - 2x)^2 + 3(x^2 - 2x) - 1$$
$$= x^6 - 6x^5 + 12x^4 - 8x^3 - 3(x^4 - 4x^3 + 4x^2) + 3(x^2 - 2x) - 1$$
$$= x^6 - 6x^5 + 12x^4 - 8x^3 - 3x^4 + 12x^3 - 12x^2 + 3x^2 - 6x - 1$$
$$= x^6 - 6x^5 + 9x^4 + 4x^3 - 9x^2 - 6x - 1, \textit{ Ans.}$$

Cube each of the following:

16. $a + b - c$.
17. $x^2 + x + 1$.
18. $x - y + 2z$.
19. $a^2 - 3a - 1$.
20. $2x^2 + x - 3$.
21. $3 - 4x + x^2$.

XIX. EVOLUTION.

189. If an expression when raised to the nth power, n being a positive integer, is equal to another expression, the first expression is said to be the nth **Root** of the second.

Thus, if $a^n = b$, a is the nth root of b.

190. **Evolution** is the process of finding any required root of an expression.

191. The **Radical Sign,** $\sqrt{\ }$, when written before an expression, indicates some root of the expression.

Thus, $\sqrt{a}$ indicates the *second,* or *square* root of a;

$\sqrt[3]{a}$ indicates the *third,* or *cube* root of a;

$\sqrt[4]{a}$ indicates the *fourth* root of a; and so on.

The *index* of a root is the number written over the radical sign to indicate what root of the expression is taken.

If no index is expressed, the index 2 is understood.

EVOLUTION OF MONOMIALS

192. **1.** Required the cube root of $a^3b^9c^6$.

We have, $(ab^3c^2)^3 = a^3b^9c^6$.

Whence, $\sqrt[3]{a^3b^9c^6} = ab^3c^2$. (§ 189)

2. Required the fifth root of $-32a^5$.

We have, $(-2a)^5 = -32a^5$.

Whence, $\sqrt[5]{-32a^5} = -2a$.

3. Required the fourth root of a^4.

We have either $(+a)^4$ or $(-a)^4$ equal to a^4.

Whence, $\sqrt[4]{a^4} = \pm a$.

The sign $\pm$, called the *double sign,* is prefixed to an expression when we wish to indicate that it is either $+$ or $-$.

193. From § 192 we derive the following rule:

Extract the required root of the absolute value of the numerical coefficient, and divide the exponent of each letter by the index of the required root.

Give to every even root of a positive term the sign ±, and to every odd root of any term the sign of the term itself.

EXAMPLES.

1. Find the square root of $9a^4b^6c^{10}$.

By the rule, $\sqrt{9a^4b^6c^{10}} = \pm 3a^2b^3c^5$, *Ans.*

2. Find the cube root of $-64x^6y^{3n}$.

$$\sqrt[3]{-64x^6y^{3n}} = -4x^2y^n, \text{ Ans.}$$

Find the values of the following:

3. $\sqrt{49a^6b^2}$.
4. $\sqrt[3]{125x^6y^9}$.
5. $\sqrt[7]{-m^{14}n^7p^{21}}$.
6. $\sqrt[4]{16a^8m^{20}}$.
7. $\sqrt{64x^{18}y^{16}z^{14}}$.
8. $\sqrt[9]{-a^9b^{18}c^{36}}$.
9. $\sqrt[8]{x^8y^{24}z^{16}}$.
10. $\sqrt[5]{243a^{25}b^{20}}$.
11. $\sqrt[3]{-512m^{15}n^{21}p^{18}}$.
12. $\sqrt{144a^{4r}x^{2s+6}}$.
13. $\sqrt[3]{-729x^{3m-6}y^{3n}}$.
14. $\sqrt[4]{256a^{8m}b^{12n}}$.

To find any root of a fraction, *extract the required root of both numerator and denominator, and divide the first result by the second.*

15. Find the value of $\sqrt[3]{-\dfrac{27a^3b^6}{8c^9}}$.

We have, $\sqrt[3]{-\dfrac{27a^3b^6}{8c^9}} = -\dfrac{\sqrt[3]{27a^3b^6}}{\sqrt[3]{8c^9}} = -\dfrac{3ab^2}{2c^3}$, *Ans.*

Find the values of the following:

16. $\sqrt{\dfrac{16a^{12}}{81b^4c^{10}}}$.
17. $\sqrt[3]{\dfrac{343x^3y^{12}}{64}}$.
18. $\sqrt[4]{\dfrac{81m^4x^{12}}{n^{16}}}$.
19. $\sqrt[5]{-\dfrac{x^5y^{10}}{32z^{15}}}$.
20. $\sqrt[6]{\dfrac{64m^6}{n^{12}}}$.
21. $\sqrt[7]{\dfrac{a^{7m}}{128b^{14n}}}$.

The root of a large number may sometimes be found by resolving it into its prime factors.

22. Find the square root of 254016.

We have, $\sqrt{254016} = \sqrt{2^6 \times 3^4 \times 7^2} = 2^3 \times 3^2 \times 7 = 504$, *Ans.*

23. Find the value of $\sqrt[3]{72 \times 75 \times 135}$.

$$\text{We have, } \sqrt[3]{72 \times 75 \times 135} = \sqrt[3]{(2^3 \times 3^2) \times (3 \times 5^2) \times (3^3 \times 5)}$$
$$= \sqrt[3]{2^3 \times 3^6 \times 5^3} = 2 \times 3^2 \times 5 = 90, \textit{ Ans.}$$

Find the values of the following:

24. $\sqrt{3136}$. **26.** $\sqrt{63504}$. **28.** $\sqrt{42 \times 56 \times 147}$.

25. $\sqrt{18225}$. **27.** $\sqrt{48 \times 54 \times 72}$. **29.** $\sqrt[3]{13824}$.

30. $\sqrt{15ab \times 21bc \times 35ca}$. **31.** $\sqrt{213444}$.

32. $\sqrt[3]{91125}$. **33.** $\sqrt[4]{20736}$. **34.** $\sqrt[5]{7776}$.

35. $\sqrt[3]{63 \times 162 \times 196}$. **36.** $\sqrt[4]{56 \times 98 \times 112}$.

37. $\sqrt{(a^2 + 5a + 6)(a^2 + 2a - 3)(a^2 + a - 2)}$.

SQUARE ROOT OF A POLYNOMIAL.

194. Since $(a + b)^2 = a^2 + 2ab + b^2$, we know that the square root of $a^2 + 2ab + b^2$ is $a + b$.

It is required to find a process by which, when the expression $a^2 + 2ab + b^2$ is given, its square root may be determined.

$$\begin{array}{r|l|l}
 & a^2 + 2ab + b^2 & a + b \\
 & a^2 & \\ \hline
2a + b & 2ab + b^2 & \\
 & 2ab + b^2 & \\ \cline{2-2}
\end{array}$$

The first term of the root, a, is found by taking the square root of the first term of the given expression.

Subtracting the square of a from the given expression, the remainder is $2ab + b^2$, or $(2a + b)b$.

If we divide the first term of this remainder by $2a$, that is, by twice the first term of the root, we obtain the second term of the root, b.

Adding this to $2a$, we obtain the complete divisor, $2a + b$.

Multiplying this by b, and subtracting the product, $2ab + b^2$, from the remainder, there are no terms remaining.

From the above process, we derive the following rule:

Arrange the expression according to the powers of some letter.

Extract the square root of the first term, write the result as the first term of the root, and subtract its square from the given expression, arranging the remainder in the same order of powers as the given expression.

Divide the first term of the remainder by twice the first term of the root, and add the quotient to the part of the root already found, and also to the trial-divisor.

Multiply the complete divisor by the term of the root last obtained, and subtract the product from the remainder.

If other terms remain, proceed as before, doubling the part of the root already found for the next trial-divisor.

EXAMPLES.

195. 1. Find the square root of $9x^4 - 30a^3x^2 + 25a^6$.

$$\begin{array}{r|r|l}
\multicolumn{2}{r|}{9x^4 - 30a^3x^2 + 25a^6} & 3x^2 - 5a^3,\ \textit{Ans.}\\
\multicolumn{2}{r|}{9x^4} & \\ \hline
6x^2 - 5a^3 & -30a^3x^2 + 25a^6 & \\
 & -30a^3x^2 + 25a^6 & \\ \cline{2-2}
\end{array}$$

The first term of the root is the square root of $9x^4$ or $3x^2$.

Subtracting the square of $3x^2$, or $9x^4$, from the given expression, the first term of the remainder is $-30a^3x^2$.

Dividing this by twice the first term of the root, or $6x^2$, we obtain the second term of the root, $-5a^3$.

Adding this to $6x^2$, we have the complete divisor, $6x^2 - 5a^3$.

Multiplying this complete divisor by $-5a^3$, and subtracting the product from the remainder, there is no remainder.

Hence, $3x^2 - 5a^3$ is the required square root.

2. Find the square root of

$$12x^5 - 22x^3 + 1 - 20x^4 + 9x^6 + 8x + 12x^2.$$

Arranging according to the descending powers of x, we have:

$$
\begin{array}{l|l|l}
 & 9x^6+12x^5-20x^4-22x^3+12x^2+8x+1 & 3x^3+2x^2-4x-1,\ \textit{Ans.} \\
 & 9x^6 & \\
\hline
6x^3+2x^2 & 12x^3 & \\
 & 12x^5+\ 4x^4 & \\
\hline
6x^3+4x^2-4x & -24x^4 & \\
 & -24x^4-16x^3+16x^2 & \\
\hline
6x^3+4x^2-8x-1 & -\ 6x^3-\ 4x^2 & \\
 & -\ 6x^3-\ 4x^2+8x+1 & \\
\hline
\end{array}
$$

It will be observed that *each trial divisor is equal to the preceding complete divisor, with its last term doubled.*

To avoid needless repetition, the last five terms of the first remainder, the last four terms of the second remainder, and the last two terms of the third-remainder are omitted.

Note. Since every square root has the double sign (§ 192), the result may be written in a different form by changing the sign of each term.

Thus, in Ex. 2, the answer may be written $1+4x-2x^2-3x^3$.

Find the square roots of the following:

3. $4x^4+4x^3+5x^2+2x+1$.
4. $1-6a+11a^2-6a^3+a^4$.
5. $9x^4-24x^3+4x^2+16x+4$.
6. $20x^3-70x+4x^4+49-3x^2$.
7. $a^2+b^2+c^2-2ab-2ac+2bc$.
8. $9a^4+1-4a^3+4a^6-6a^2+12a^5$.
9. $x^6-4x^4a^2+10x^3a^3+4x^2a^4-20xa^5+25a^6$.
10. $9x^2+25y^2+16z^2+30xy-24xz-40yz$.
11. $49m^4-14m^3n-55m^2n^2+8mn^3+16n^4$.
12. $49a^2-30x^3+16+9a^4-40a$.
13. $25x^4-20x^3y-26x^2y^2+12xy^3+9y^4$.
14. $16m^4+8m^3x^2-23m^2x^4-6mx^6+9x^8$.

15. $20\,ab^3 + 9\,a^4 - 26\,a^2b^2 + 25\,b^4 - 12\,a^3b.$

16. $m^2 + 8\,m + 12 - \frac{16}{m} + \frac{4}{m^2}.$

17. $1 - 2\,x + 3\,x^2 - 4\,x^3 + 3\,x^4 - 2\,x^5 + x^6.$

18. $12\,x^4 + 12\,x - 8\,x^5 + 9 + 28\,x^2 + x^6 + 10\,x^3.$

19. $x^2 - xy - \frac{11\,y^2}{4} + \frac{3\,y^3}{2\,x} + \frac{9\,y^4}{4\,x^2}.$

20. $\frac{x^4}{9} - \frac{x^3}{3} + \frac{31\,x^2}{60} - \frac{2\,x}{5} + \frac{4}{25}.$

21. $4\,a^6 + 12\,a^5b + 25\,a^4b^2 + 4\,a^3b^3 - 14\,a^2b^4 - 40\,ab^5 + 25\,b^6.$

22. $\frac{a^4}{4} + \frac{a^3b}{3} + \frac{13\,a^2b^2}{36} + \frac{ab^3}{6} + \frac{b^4}{16}.$

23. $28\,x^3y^3 + 9\,x^6 - 15\,x^2y^4 - 12\,x^5y - 8\,xy^5 - 2\,x^4y^2 + 16\,y^6.$

24. $\frac{16}{9} + \frac{8\,x}{3\,a} - \frac{13\,x^2}{3\,a^2} - \frac{4\,x^3}{a^3} + \frac{4\,x^4}{a^4}.$

Find to four terms the approximate square roots of:

25. $1 + 4\,x.$

26. $1 + 2\,a.$

27. $1 - x.$

28. $1 - 3\,a.$

29. $x^2 + 6.$

30. $4\,a^2 - 2\,b.$

SQUARE ROOT OF AN ARITHMETICAL NUMBER.

196. The square root of 100 is 10; of 10000 is 100; etc.

Hence, the square root of a number between 1 and 100 is between 1 and 10; the square root of a number between 100 and 10000 is between 10 and 100; etc.

That is, the integral part of the square root of a number of one or two figures, contains *one* figure; of a number of three or four figures, contains *two* figures; and so on.

Hence, *if a point be placed over every second figure of any integral number, beginning with the units' place, the number of points shows the number of figures in the integral part of its square root.*

197. Let it be required to find the square root of 4624.

$$\begin{array}{r|l|l}
\multicolumn{2}{r|}{a^2 + 2\,ab + b^2 = 4\dot{6}2\dot{4}} & 60 + 8 \\
\multicolumn{2}{r|}{a^2 = 3600} & = a + b \\
\cline{1-3}
120 + 8 & 1024 = 2\,ab + b^2 & \\
= 2\,a + b & 1024 & \\
\cline{2-2}
\end{array}$$

Pointing the number according to the rule of § 196, we find that there are two figures in the integral part of its square root.

Let a denote the greatest multiple of 10 whose square is less than 4624; this we find by inspection to be 60.

Let b denote the digit in the units' place of the root; then, the given number is denoted by $(a + b)^2$, or $a^2 + 2\,ab + b^2$.

Subtracting a^2, or 3600, from 4624, the remainder is 1024.

That is, $$2\,ab + b^2 = 1024. \qquad (1)$$

Since b^2 is generally small in comparison with $2\,ab$, we may obtain an *approximate* value of b by neglecting the b^2 term in (1).

Then, $2\,ab = 1024$, and $b = \frac{1024}{2\,a} = \frac{1024}{120} = 8 +$.

This suggests that the digit in the units' place is 8.

If this be correct, $2\,ab + b^2$, or $(2\,a + b)b$, must equal 1024.

Adding 8 to 120, multiplying the sum by 8, and subtracting the product from 1024, there is no remainder.

Hence, $60 + 8$, or 68, is the required square root.

Omitting the ciphers, for the sake of brevity, and condensing the operation, it will stand as follows:

$$\begin{array}{r|l|l}
\multicolumn{2}{r|}{4\dot{6}2\dot{4}} & 68 \\
\multicolumn{2}{r|}{36} & \\
\cline{1-3}
128 & 1024 & \\
 & 1024 & \\
\cline{2-2}
\end{array}$$

From the above example, we derive the following rule:

Separate the number into periods by pointing every second figure, beginning with the units' place.

Find the greatest square in the left-hand period, and write its square root as the first figure of the root; subtract the square of the first root-figure from the left-hand period, and to the result annex the next period.

Divide this remainder, omitting the last figure, by twice the part of the root already found, and annex the quotient to the root, and also to the trial-divisor.

Multiply the complete divisor by the root-figure last obtained, and subtract the product from the remainder.

If other periods remain, proceed as before, doubling the part of the root already found for the next trial-divisor.

Note 1. It sometimes happens that, on multiplying a complete divisor by the figure of the root last obtained, the product is greater than the remainder.

In such a case, the figure of the root last obtained is too great, and one less must be substituted for it.

Note 2. If any root-figure is 0, annex 0 to the trial-divisor, and annex to the remainder the next period.

198. Required the square root of 4944.9024.

We have, $\sqrt{4944.9024} = \sqrt{\frac{49449024}{10000}} = \frac{\sqrt{49449024}}{\sqrt{10000}}.$

$$
\begin{array}{r|l|l}
\multicolumn{2}{r|}{4\dot{9}4\dot{4}9\dot{0}2\dot{4}} & 7032 \\
\multicolumn{2}{l|}{\quad 49} & \\
\cline{1-2}
1403 & 4490 & \\
 & 4209 & \\
\cline{1-2}
14062 & 28124 & \\
 & 28124 & \\
\cline{2-2}
\end{array}
$$

Since 14 is not contained in 4, we write 0 as the second root-figure, annex 0 to the trial-divisor 14, and annex to the remainder the next period, 90. (See Note 2, § 197.)

Then, $\sqrt{4944.9024} = \frac{7032}{100} = 70.32.$

The work may be arranged as follows:

$$
\begin{array}{r|l|l}
\multicolumn{2}{r|}{4\dot{9}4\dot{4}.9\dot{0}2\dot{4}} & 70.32 \\
\multicolumn{2}{l|}{\quad 49} & \\
\cline{1-2}
1403 & 44\;90 & \\
 & 42\;09 & \\
\cline{1-2}
14062 & 2\;8124 & \\
 & 2\;8124 & \\
\cline{2-2}
\end{array}
$$

It follows from the above that, *if a point be placed over every second figure of any number, beginning with the units' place, and extending in either direction, the rule of* § 197 *may be applied to the result and the decimal point inserted in its proper position in the root.*

EXAMPLES.

199. Find the square roots of the following:

1. 4225.
2. 21904.
3. 508369.
4. 65.1249.
5. .156816.
6. .064516.
7. 3956.41.
8. 96.4324.
9. .00321489.
10. 12823561.
11. 75570.01.
12. .16216729.
13. 2666.6896.
14. .0062504836.
15. 86.825124.

If there is a final remainder, the number has no exact square root; but we may continue the operation by annexing periods of ciphers, and thus obtain an approximate root, correct to any desired number of decimal places.

16. Find the square root of 12 to four decimal places.

$$\begin{array}{r|l}
 & 1\dot{2}.0\dot{0}0\dot{0}0\dot{0}0\dot{0} \;\big|\; 3.4641+, \textit{ Ans.} \\
 & 9 \\ \hline
64 & 3\,00 \\
 & 2\,56 \\ \hline
686 & 4400 \\
 & 4116 \\ \hline
6924 & 28400 \\
 & 27696 \\ \hline
69281 & 70400
\end{array}$$

Find the first five figures of the square root of:

17. 7.
18. 8.
19. 10.
20. 13.
21. 48.
22. 64.7.
23. .2.
24. .056.
25. .39.
26. .009.
27. .00074.
28. 8.5645.

The square root of a fraction may be obtained by taking the square root of the numerator, and then of the denominator, and dividing the first result by the second.

If the denominator is not a perfect square, it is better to reduce the fraction to an equivalent fraction whose denominator is a perfect square.

29. Find the value of $\sqrt{\frac{3}{8}}$ to five decimal places.

We have, $\sqrt{\frac{3}{8}}=\sqrt{\frac{6}{16}}=\frac{\sqrt{6}}{\sqrt{16}}=\frac{2.44948+}{4}=.61237+$, *Ans.*

Find the first four figures of the square root of:

30. $\frac{3}{4}$.

31. $\frac{5}{9}$.

32. $\frac{33}{25}$.

33. $\frac{1}{2}$.

34. $\frac{3}{5}$.

35. $\frac{7}{8}$.

36. $\frac{13}{32}$.

37. $\frac{15}{14}$.

38. $\frac{19}{12}$.

39. $\frac{10}{27}$.

CUBE ROOT OF A POLYNOMIAL.

200. Since $(a+b)^3=a^3+3a^2b+3ab^2+b^3$, we know that the cube root of $a^3+3a^2b+3ab^2+b^3$ is $a+b$.

It is required to find a process by which, when the expression $a^3+3a^2b+3ab^2+b^3$ is given, its cube root may be determined.

$$\begin{array}{r|l|l}
 & a^3+3a^2b+3ab^2+b^3 & a+b \\
 & a^3 & \\
\hline
3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 & \\
 & 3a^2b+3ab^2+b^3 & \\
\cline{2-2}
\end{array}$$

The first term of the root, a, is found by taking the cube root of the first term of the given expression.

Subtracting the cube of a from the given expression, the remainder is $3a^2b+3ab^2+b^3$, or $(3a^2+3ab+b^2)b$.

If we divide the first term of this remainder by $3a^2$, that is, by three times the square of the first term of the root, we obtain the second term of the root, b.

Adding to the trial-divisor $3ab$, that is, three times the product of the first term of the root by the second, and b^2, that is, the square of the second term of the root, we obtain the complete divisor, $3a^2 + 3ab + b^2$.

Multiplying this by b, and subtracting the product, $3a^2b+3ab^2+b^3$ from the remainder, there are no terms remaining.

From the above process, we derive the following rule:

Arrange the expression according to the powers of some letter.

Extract the cube root of the first term, write the result as the first term of the root, and subtract its cube from the given expression; arranging the remainder in the same order of powers as the given expression.

Divide the first term of the remainder by three times the square of the first term of the root, and write the result as the next term of the root.

Add to the trial-divisor three times the product of the term of the root last obtained by the part of the root previously found, and the square of the term of the root last obtained.

Multiply the complete divisor by the term of the root last obtained, and subtract the product from the remainder.

If other terms remain, proceed as before, taking three times the square of the part of the root already found for the next trial-divisor.

EXAMPLES.

201. 1. Find the cube root of

$$8x^6 - 36x^4y + 54x^2y^2 - 27y^3.$$

$$\begin{array}{r|l|l}
 & 8x^6 - 36x^4y + 54x^2y^2 - 27y^3 & 2x^2 - 3y, \textit{ Ans.} \\
 & 8x^6 & \\
\hline
12x^4 - 18x^2y + 9y^2 & -36x^4y + 54x^2y^2 - 27y^3 & \\
 & -36x^4y + 54x^2y^2 - 27y^3 & \\
\hline
\end{array}$$

The first term of the root is the cube root of $8x^6$, or $2x^2$.

Subtracting the cube of $2x^2$, or $8x^6$, from the given expression, the first term of the remainder is $-36x^4y$.

Dividing this by three times the square of the first term of the root, or $12x^4$, we obtain the second term of the root, $-3y$.

Adding to the trial-divisor three times the product of the term of the root last obtained by the part of the root previously found, or $-18x^2y$, and the square of the term of the root last obtained, or $9y^2$, we have the complete divisor, $12x^4 - 18x^2y + 9y^2$.

Multiplying this complete divisor by $-3y$, and subtracting the product from the remainder, there is no remainder.

Hence, $2x^2 - 3y$ is the required cube root.

2. Find the cube root of

$$28x^3 - 54x + x^6 + 3x^4 - 9x^2 - 27 - 6x^5.$$

Arranging according to the descending powers of x, we have

$$\begin{array}{r|l|l}
 & x^6-6x^5+3x^4+28x^3-9x^2-54x-27 & x^2-2x-3, \\
 & x^6 & \\ \hline
3x^4-6x^3+4x^2 & -6x^5 & \\
 & -6x^5+12x^4-8x^3 & \\ \hline
3x^4-12x^3+12x^2 & -9x^4+36x^3 & \\
-9x^2+18x+9 & & \\ \hline
3x^4-12x^3+3x^2+18x+9 & -9x^4+36x^3-9x^2-54x-27 & \\ \hline
\end{array}$$

The second complete divisor is formed as follows:

The trial-divisor is three times the square of the part of the root already found; that is, $3(x^2-2x)^2$, or $3x^4 - 12x^3 + 12x^2$.

Three times the product of the term of the root last obtained by the part of the root previously found is $3(-3)(x^2-2x)$, or $-9x^2 + 18x$.

The square of the term of the root last obtained is $(-3)^2$, or 9.

Adding these, the complete divisor is $3x^4 - 12x^3 + 3x^2 + 18x + 9$.

The last five terms of the first remainder and the last three terms of the second remainder are omitted.

Find the cube roots of the following:

3. $8x^3 + 12x^2 + 6x + 1$.

4. $1 - 12a^3 + 48a^6 - 64a^9$.

5. $27m^6 + 135m^4n + 225m^2n^2 + 125n^3$.

6. $294ab^2 - 84a^2b - 343b^3 + 8a^3$.

7. $x^6 - 6x^5 + 9x^4 + 4x^3 - 9x^2 - 6x - 1$.

8. $8a^6 + 36a^5 + 66a^4 + 63a^3 + 33a^2 + 9a + 1.$

9. $30y^2 + 27y^6 + 12y - 45y^4 - 8 - 35y^3 + 27y^5.$

10. $\frac{a^3}{8} - \frac{a^2b}{4} + \frac{ab^2}{6} - \frac{b^3}{27}.$

11. $9a^3 - 36a + a^6 + 21a^4 - 9a^5 - 8 - 42a^2.$

12. $174x^4 + 8 + 174x^2 - 60x^5 - 245x^3 + 8x^6 - 60x.$

13. $27a^6 - 54a^5b + 63a^4b^2 - 44a^3b^3 + 21a^2b^4 - 6ab^5 + b^6.$

14. $6x^5y + 96xy^5 + 56x^3y^3 + x^6 + 24x^4y^2 + 64y^6 + 96x^2y^4.$

15. $\frac{x^3}{27} - \frac{x^2}{3} + \frac{7x}{3} - 9 + \frac{28}{x} - \frac{48}{x^2} + \frac{64}{x^3}.$

CUBE ROOT OF AN ARITHMETICAL NUMBER.

202. The cube root of 1000 is 10; of 1000000 is 100; etc. Hence, the cube root of a number between 1 and 1000 is between 1 and 10; the cube root of a number between 1000 and 1000000 is between 10 and 100; etc.

That is, the integral part of the cube root of a number of one, two, or three figures, contains *one* figure; of a number of four, five, or six figures, contains *two* figures; and so on.

Hence, *if a point be placed over every third figure of any integral number, beginning with the units' place, the number of points shows the number of figures in the integral part of its cube root.*

203. Let it be required to find the cube root of 157464.

$$\begin{array}{r|l} a^3 + 3a^2b + 3ab^2 + b^3 = 15\dot{7}46\dot{4} & 50 + 4 = a + b \\ a^3 = 125000 & \\ \hline \end{array}$$

$$\begin{array}{r|l} 3a^2 = 7500 & 32464 = 3a^2b + 3ab^2 + b^3 \\ 3ab = 600 & \\ b^2 = \underline{16} & \\ 3a^2 + 3ab + b^2 = 8116 & \underline{32464} \end{array}$$

Pointing the number according to the rule of § 202, we find that there are two figures in the integral part of its cube root.

Let a denote the greatest multiple of 10 whose cube is less than 157464; this we find, by inspection, to be 50.

Let b denote the digit in the units' place of the root; then, the given number is denoted by $(a + b)^3$, or $a^3 + 3a^2b + 3ab^2 + b^3$.

Subtracting a^3, or 125000, from 157464, the remainder is 32464.

That is, $$3a^2b + 3ab^2 + b^3 = 32464. \qquad (1)$$

Since $3ab^2$ and b^3 are generally small in comparison with $3a^2b$, we may obtain an *approximate* value of b by neglecting the $3ab^2$ and b^3 terms in (1).

Then, $3a^2b = 32464$, and $b = \frac{32464}{3a^2} = \frac{32464}{7500} = 4 +$.

This suggests that the digit in the units' place is 4.

If this be correct, $3a^2b + 3ab^2 + b^3$, or $(3a^2 + 3ab + b^2)b$, must equal 32464.

Adding to 7500 $3ab$, or 600, and b^2, or 16, the sum is 8116; multiplying this by 4, and subtracting the product from 32464, there is no remainder.

Hence, 50 + 4, or 54, is the required cube root.

Omitting the ciphers for the sake of brevity, and condensing the operation, it will stand as follows:

$$\begin{array}{r|r|l}
 & 15\dot{7}46\dot{4} & 54 \\
 & 125 & \\
\hline
7500 & 32464 & \\
600 & & \\
16 & & \\
\hline
8116 & 32464 & \\
\hline
\end{array}$$

From the above example, we derive the following rule:

Separate the number into periods by pointing every third figure, beginning with the units' place.

Find the greatest cube in the left-hand period, and write its cube root as the first figure of the root; subtract the cube of the first root-figure from the left-hand period, and to the result annex the next period.

Divide this remainder by three times the square of the part of the root already found, with two ciphers annexed, and write the quotient as the next figure of the root.

Add to the trial-divisor three times the product of the last root-figure by the part of the root previously found, with one cipher annexed, and the square of the last root-figure.

Multiply the complete divisor by the figure of the root last obtained, and subtract the product from the remainder.

If other periods remain, proceed as before, taking three times the square of the part of the root already found, with two ciphers annexed, as the next trial-divisor.

Note 1. Note 1, p. 181, applies with equal force to the above rule.

Note 2. If any root-figure is 0, annex two ciphers to the trial-divisor, and annex to the remainder the next period.

204. If, in the example of § 203, there had been more periods in the given number, the next trial-divisor would have been three times the square of $a+b$, or $3a^2+6ab+3b^2$.

We observe that this may be obtained from the preceding complete divisor, $3a^2+3ab+b^2$, by adding to it its second term, $3ab$, and twice its third term, $2b^2$.

Hence, *if the first number and twice the second number required to complete any trial-divisor, be added to the complete divisor, the result, with two ciphers annexed, will be the next trial-divisor.*

205. Required the cube root of 8144.865728.

We have, $\sqrt[3]{8144.865728}=\sqrt[3]{\dfrac{8144865728}{1000000}}=\dfrac{\sqrt[3]{8144865728}}{\sqrt[3]{1000000}}.$

$$\begin{array}{r|l|l}
 & \dot{8}14\dot{4}86\dot{5}72\dot{8} & 2012 \\
 & 8 & \\ \hline
120000 & 144865 & \\
600 & & \\
1 & & \\ \hline
120601 & 120601 & \\ \hline
600 & 24264728 & \\
2 & & \\ \hline
12120300 & & \\
12060 & & \\
4 & & \\ \hline
12132364 & 24264728 & \\ \hline
\end{array}$$

Since 1200 is not contained in 144, the second root-figure is 0; we then annex two ciphers to the trial-divisor 1200, and annex to the remainder the next period, 865.

The second trial-divisor is formed by the rule of § 204. Adding to the complete divisor 120601 the first number, 600, and twice the second number, 2, required to complete the trial-divisor 120000, we have 121203; annexing two ciphers to this, the result is 12120300.

Then, $$\sqrt[3]{8144.865728} = \frac{2012}{100} = 20.12.$$

The work may be arranged as follows:

	$\dot{8}14\dot{4}.86\dot{5}72\dot{8}$	20.12
	8	
120000	144 865	
600		
1		
120601	120 601	
600	24 264728	
2		
12120300		
12060		
4		
12132364	24 264728	

It follows from the above that, *if a point be placed over every third figure of any number, beginning with the units' place, and extending in either direction, the rule of* § 203 *may be applied to the result, and the decimal point inserted in its proper position in the root.*

EXAMPLES.

206. Find the cube roots of the following:

1. 19683.
2. 148877.
3. 59.319.
4. .614125.
5. 2515456.
6. 857.375.
7. .224755712.
8. 46.268279.
9. 523606616.
10. 187149.248.
11. .000111284641.
12. 788889.024.
13. 444.194947.
14. 338608873.
15. .001151022592.

Find the first four figures of the cube root of:

16. 3. **18.** 9.1. **20.** $\frac{5}{8}$. **22.** $\frac{4}{9}$.

17. 7. **19.** .02. **21.** $\frac{11}{27}$. **23.** $\frac{2}{5}$.

207. If the index of the required root is the product of two or more numbers, we may obtain the result by *successive extractions of the simpler roots.*

For by § 189, $(\sqrt[mn]{a})^{mn} = a.$

Taking the nth root of both members,

$$(\sqrt[mn]{a})^{m} = \sqrt[n]{a}. \qquad (1)$$

Taking the mth root of both members of (1),

$$\sqrt[mn]{a} = \sqrt[m]{\sqrt[n]{a}}.$$

Hence, *the* mnth *root of an expression is equal to the* mth *root of the* nth *root of the expression.*

Thus, the fourth root is the square root of the square root; the sixth root is the cube root of the square root, etc.

EXAMPLES.

Find the fourth roots of the following:

1. $81a^4 + 216a^3b^2 + 216a^2b^4 + 96ab^6 + 16b^8.$

2. $1 - 12x + 50x^2 - 72x^3 - 21x^4 + 72x^5 + 50x^6 + 12x^7 + x^8.$

3. $16a^8 - 32a^7 - 40a^6 + 88a^5 + 49a^4 - 88a^3 - 40a^2 + 32a + 16.$

Find the sixth roots of the following:

4. $x^{18} + 6x^{15}y^2 + 15x^{12}y^4 + 20x^9y^6 + 15x^6y^8 + 6x^3y^{10} + y^{12}.$

5. $a^6 - 12a^5 + 60a^4 - 160a^3 + 240a^2 - 192a + 64.$

6. Find the fourth root of 209727.3616. 21.4

7. Find the sixth root of .009474296896.

XX. THEORY OF EXPONENTS.

208. In the preceding chapters, an exponent has been considered only as a *positive integer*.

Thus, if m is a positive integer,

$$a^m = a \times a \times a \times \cdots \text{ to } m \text{ factors.} \qquad (\S\ 6)$$

209. Let m and n be positive integers.

Then, $a^m = a \times a \times a \times \cdots$ to m factors,

and $a^n = a \times a \times a \times \cdots$ to n factors.

Whence, $a^m \times a^n = a \times a \times a \times \cdots$ to $m + n$ factors.

That is, $a^m \times a^n = a^{m+n}$. (1)

This proves the law stated in § 46 for all positive integral values of the exponents.

Again,

$$(a^m)^n = a^m \times a^m \times a^m \times \cdots \text{ to } n \text{ factors}$$
$$= a^{m+m+m+\cdots \text{ to } n \text{ terms}}.$$

That is, $(a^m)^n = a^{mn}$. (2)

This proves the first paragraph of the law stated in § 186 for all positive integral values of the exponents.

210. It is found convenient to employ exponents which are not positive integers; and we proceed to define them, and to prove the rules for their use.

It will be convenient to have all forms of exponents subject to the same laws in regard to multiplication, division, etc.; and we shall therefore find what meanings must be attached to fractional, negative, and zero exponents in order that equation (1), § 209, may hold for *all* values of m and n.

211. Meaning of a Fractional Exponent.

1. Required the meaning of $a^{\frac{5}{3}}$.

If (1), § 209, is to hold for all values of m and n, we have

$$a^{\frac{5}{3}} \times a^{\frac{5}{3}} \times a^{\frac{5}{3}} = a^{\frac{5}{3}+\frac{5}{3}+\frac{5}{3}} = a^5.$$

Hence, $a^{\frac{5}{3}}$ is such an expression that its third power is a^5.

Then, $a^{\frac{5}{3}}$ must be the *cube root* of a^5; or, $a^{\frac{5}{3}} = \sqrt[3]{a^5}$.

2. Required the meaning of $a^{\frac{p}{q}}$, where p and q are any positive integers.

If (1), § 209, is to hold for all values of m and n, we have

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \cdots \text{ to } q \text{ factors} = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\cdots \text{ to } q \text{ terms}} = a^{\frac{p}{q}\times q} = a^p.$$

Hence, $a^{\frac{p}{q}}$ is such an expression that its qth power is a^p.

Then, $a^{\frac{p}{q}}$ must be the qth root of a^p; or, $a^{\frac{p}{q}} = \sqrt[q]{a^p}$.

Hence, *in a fractional exponent, the numerator denotes a power, and the denominator a root.*

For example, $a^{\frac{3}{4}} = \sqrt[4]{a^3}$; $b^{\frac{5}{2}} = \sqrt{b^5}$; $x^{\frac{1}{3}} = \sqrt[3]{x}$; etc.

EXAMPLES.

212. Express the following with radical signs:

1. $a^{\frac{2}{7}}$. 3. $4x^{\frac{3}{2}}$. 5. $a^{\frac{4}{9}}x^{\frac{3}{4}}$. 7. $6x^{\frac{4}{5}}y^{\frac{1}{6}}$. 9. $ab^{\frac{5}{8}}c^{\frac{3}{7}}d^{\frac{7}{6}}$.

2. $b^{\frac{1}{5}}$. 4. $9ab^{\frac{7}{3}}$. 6. $m^{\frac{1}{2}}n^{\frac{5}{4}}$. 8. $8a^{\frac{1}{6}}m^{\frac{6}{5}}$. 10. $3x^{\frac{m}{n}}y^{\frac{1}{p}}z^{\frac{q}{2}}$.

Express the following with fractional exponents:

11. $\sqrt[5]{a^3}$. 13. $\sqrt{m^7}$. 15. $2\sqrt[3]{n^8}$. 17. $\sqrt[4]{a^7}\sqrt[7]{b^4}$.

12. $\sqrt[4]{x}$. 14. $\sqrt[8]{b^9}$. 16. $5\sqrt[6]{y^5}$. 18. $\sqrt[3]{m^2}\sqrt[5]{n^6}$.

19. $7\sqrt[9]{x}\sqrt{y^9}$. 20. $\sqrt[r]{a^s}\sqrt[m]{b}\sqrt{c^n}$.

213. Meaning of a Zero Exponent.

If (1), § 209, is to hold for all values of m and n, we have

$$a^m \times a^0 = a^{m+0} = a^m.$$

Whence, $$a^0 = \frac{a^m}{a^m} = 1.$$

Hence, *the zero power of any quantity is equal to* **1**.

214. Meaning of a Negative Exponent.

1. Required the meaning of a^{-3}.

If (1), § 209, is to hold for all values of m and n, we have

$$a^{-3} \times a^3 = a^{-3+3} = a^0 = 1. \qquad (§\ 213)$$

Whence, $$a^{-3} = \frac{1}{a^3}.$$

2. Required the meaning of a^{-s}, where s is a positive integer or a positive fraction.

If (1), § 209, is to hold for all values of m and n, we have

$$a^{-s} \times a^s = a^{-s+s} = a^0 = 1. \qquad (§\ 213)$$

Whence, $$a^{-s} = \frac{1}{a^s}.$$

For example, $a^{-2} = \frac{1}{a^2}$; $a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}}$; $3x^{-1}y^{-\frac{1}{2}} = \frac{3}{xy^{\frac{1}{2}}}$; etc.

215. It follows from § 214 that

Any factor of the numerator of a fraction may be transferred to the denominator, or any factor of the denominator to the numerator, if the sign of its exponent be changed.

Thus, $$\frac{a^2b^3}{cd^4} = \frac{b^3}{a^{-2}cd^4} = \frac{a^2b^3c^{-1}}{d^4} = \frac{a^2d^{-4}}{b^{-3}c}, \text{ etc.}$$

EXAMPLES.

216. Write the following with positive exponents:

1. a^2b^{-5}.
5. $a^{-3}x^{-4}$.
9. $6x^{-10}y^{-\frac{5}{6}}$.
2. $m^{-1}n^{\frac{4}{7}}$.
6. $5mx^{-\frac{4}{5}}$.
10. $3x^{-\frac{4}{3}}y^6z^{-\frac{7}{4}}$.
3. $x^8y^{-\frac{1}{3}}$.
7. $4a^{-6}b^{-\frac{3}{2}}$.
11. $6m^{-2}n^{-\frac{6}{7}}p^{\frac{4}{5}}$.
4. $2a^{\frac{2}{3}}n^{-9}$.
8. $a^{-\frac{7}{8}}b^{-8}c^5$.
12. $a^{-7}b^{-\frac{2}{3}}c^{-\frac{9}{10}}$.

Transfer the literal factors from the denominators to the numerators in the following:

13. $\frac{1}{a^4}$.
15. $\frac{2}{ab^{-3}}$.
17. $\frac{9x^6}{y^{-\frac{1}{4}}z^{-5}}$.
19. $\frac{6m^{\frac{1}{2}}n^{-8}}{5p^9q}$.
14. $\frac{x^{-\frac{4}{3}}}{y^{\frac{2}{3}}}$.
16. $\frac{1}{7m^{-7}x^{\frac{4}{5}}}$.
18. $\frac{a^{-2}}{4b^{-\frac{2}{9}}c}$.
20. $\frac{3a^{-\frac{3}{8}}x^{\frac{5}{4}}}{8b^{-\frac{7}{6}}y^{\frac{8}{7}}}$.

Transfer the literal factors from the numerators to the denominators in the following:

21. $\frac{3x^6}{2}$.
23. $\frac{a^2b^{\frac{5}{2}}}{3}$.
25. $\frac{7x^{-7}y}{z^{\frac{1}{6}}}$.
27. $\frac{8a^{-\frac{3}{4}}b^{-9}}{5x^3y^{-2}}$.
22. $\frac{b^{\frac{1}{3}}}{c^4}$.
24. $\frac{mn^{-8}}{6}$.
26. $\frac{a^{-\frac{2}{5}}x^{-4}}{y^{-\frac{4}{9}}}$.
28. $\frac{4a^{-\frac{6}{7}}m^{\frac{9}{8}}}{9b^{-5}n^{-\frac{5}{3}}}$.

217. Since the definitions of fractional, negative, and zero exponents were obtained on the supposition that equation (1), § 209, was to hold universally, we have for all values of m and n

$$a^m \times a^n = a^{m+n}.$$

For example, $a^2 \times a^{-5} = a^{2-5} = a^{-3}$;

$$a^{\frac{3}{4}} \times a^{-\frac{2}{3}} = a^{\frac{3}{4}-\frac{2}{3}} = a^{\frac{1}{12}};$$

$$a \times \sqrt{a^5} = a \times a^{\frac{5}{2}} = a^{1+\frac{5}{2}} = a^{\frac{7}{2}};\ \text{etc.}$$

EXAMPLES.

Find the values of the following:

$^{-2}$ 1. $a^5 \times a^{-7}$. $m^{\frac{1}{6}}$ 4. $m^{\frac{1}{2}} \times m^{-\frac{1}{3}}$. 7. $5x^{-2} \times 4x^{-\frac{4}{5}}$.

$a^0 = 1$ 2. $a^3 \times a^{-3}$. $2n$ 5. $2n^{\frac{7}{2}} \times n^{-\frac{5}{2}}$. 7 8. $m^2 \times \sqrt[4]{m}$.

x^{-9} 3. $x^{-4} \times x^{-5}$. $3a^{-\frac{3}{8}}$ 6. $a \times 3a^{-\frac{11}{8}}$. 9. $c^3 \times \sqrt[7]{c^{-2}}$.

10. $n^{-1} \times \dfrac{6}{n^{-\frac{4}{5}}}$.

11. $7\sqrt[5]{a^2} \times \sqrt{a^{-3}}$.

12. $3a\sqrt[6]{b^7} \times 2\sqrt[3]{b^4}$.

13. $x^{-6}y^{\frac{4}{7}} \times 4x^7y^{-2}$.

14. $m^4n^{-\frac{5}{4}} \times \frac{1}{5}m^{-4}n^{-\frac{1}{6}}$.

15. $\dfrac{1}{a^{-6}x^{\frac{3}{4}}} \times a^{-5}x^{-\frac{1}{3}}$.

16. Multiply $a + 2a^{\frac{2}{3}} - 3a^{\frac{1}{3}}$ by $2 - 4a^{-\frac{1}{3}} - 6a^{-\frac{2}{3}}$.

$$
\begin{array}{llll}
a + 2a^{\frac{2}{3}} & - \ 3a^{\frac{1}{3}} & & \\
2 - 4a^{-\frac{1}{3}} & - \ 6a^{-\frac{2}{3}} & & \\
\hline
2a + 4a^{\frac{2}{3}} & - \ 6a^{\frac{1}{3}} & & \\
\quad - 4a^{\frac{2}{3}} & - \ 8a^{\frac{1}{3}} + 12 & & \\
 & - \ 6a^{\frac{1}{3}} - 12 & + 18a^{-\frac{1}{3}} & \\
\hline
2a & - 20a^{\frac{1}{3}} & + 18a^{-\frac{1}{3}}, & \textit{Ans.}
\end{array}
$$

Note. It must be carefully borne in mind, in examples like the above, that the zero power of any quantity is equal to 1 (§ 213).

Multiply the following:

17. $a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.

18. $4x^{-\frac{4}{3}} - 6x^{-\frac{2}{3}} + 9$ by $2x^{-\frac{2}{3}} + 3$.

19. $2a^{-1} - 7 - 3a$ by $4a^{-1} + 5$.

20. $x^{-\frac{9}{4}} + 2x^{-\frac{3}{2}} + 4x^{-\frac{3}{4}} + 8$ by $x^{-\frac{3}{4}} - 2$.

21. $x^{\frac{1}{3}} + x^{\frac{1}{6}}y^{\frac{1}{6}} + y^{\frac{1}{3}}$ by $x^{\frac{1}{3}} - x^{\frac{1}{6}}y^{\frac{1}{6}} + y^{\frac{1}{3}}$.

22. $m - 2m^{\frac{4}{5}}n^{-\frac{1}{3}} + m^{\frac{3}{5}}n^{-\frac{2}{3}}$ by $m^{\frac{2}{5}}n^{-1} - 2m^{\frac{1}{5}}n^{-\frac{4}{3}} + n^{-\frac{5}{3}}$.

23. $a^{-2}b^{-3}+a^{-3}b^{-4}-a^{-4}b^{-5}$ by $a^{-1}b^{-2}-a^{-2}b^{-3}-a^{-3}b^{-4}$.

24. $m^{-1}+2m^{-\frac{2}{3}}n^{-1}+3m^{-\frac{1}{3}}n^{-2}$ by $2m^{-\frac{1}{3}}-4n^{-1}+6m^{\frac{1}{3}}n^{-2}$.

25. $2a^{\frac{3}{4}}b^{-2}+a^{\frac{1}{4}}-4a^{-\frac{1}{4}}b^{2}$ by $2a^{\frac{1}{2}}-b^{2}-4a^{-\frac{1}{2}}b^{4}$.

26. $3m^{\frac{3}{2}}x^{\frac{1}{3}}-4mx^{\frac{2}{3}}+m^{\frac{1}{2}}x$ by $6m^{\frac{1}{4}}x^{-\frac{2}{3}}+8m^{-\frac{1}{4}}x^{-\frac{1}{3}}+2m^{-\frac{3}{4}}$.

218. *To prove* $\dfrac{a^m}{a^n}=a^{m-n}$ *for all values of m and n.*

By § 215, $\dfrac{a^m}{a^n}=a^m\times a^{-n}=a^{m-n}$, by (1), § 209.

For example, $\dfrac{a^{-\frac{3}{4}}}{a}=a^{-\frac{3}{4}-1}=a^{-\frac{7}{4}}$;

$$\frac{a^{\frac{1}{2}}}{a^{-2}}=a^{\frac{1}{2}+2}=a^{\frac{5}{2}};$$

$$\frac{a^{-3}}{\sqrt[5]{a^{-2}}}=\frac{a^{-3}}{a^{-\frac{2}{5}}}=a^{-3+\frac{2}{5}}=a^{-\frac{13}{5}};\text{ etc.}$$

EXAMPLES.

Divide the following:

1. a^4 by a^7.
2. x by $x^{\frac{1}{3}}$.
3. $m^{\frac{1}{4}}$ by $m^{-\frac{7}{8}}$.
4. $a^{-\frac{2}{3}}$ by $a^{\frac{1}{3}}$.
5. b^{-2} by $\sqrt{b^{-5}}$.
6. $2\sqrt[6]{x}$ by $x^{-\frac{5}{3}}$.
7. n^2 by $\dfrac{1}{\sqrt[3]{n^2}}$.
8. $10a^{-3}b^{-\frac{7}{2}}$ by $5a^5b^{-3}$.
9. $6\sqrt[4]{x^{-5}}$ by $2\sqrt[5]{x^{-2}}$.
10. Divide $2a^{\frac{2}{3}}-20+18a^{-\frac{2}{3}}$ by $a+2a^{\frac{2}{3}}-3a^{\frac{1}{3}}$.

$$\begin{array}{r|l}
2a^{\frac{2}{3}}-20+18a^{-\frac{2}{3}} & a+2a^{\frac{2}{3}}-3a^{\frac{1}{3}} \\ \cline{2-2}
\underline{2a^{\frac{2}{3}}+4a^{\frac{1}{3}}-6\quad\quad\quad} & 2a^{-\frac{1}{3}}-4a^{-\frac{2}{3}}-6a^{-1},\ \textit{Ans.} \\
-4a^{\frac{1}{3}}-14+18a^{-\frac{2}{3}} & \\
\underline{-4a^{\frac{1}{3}}-\ \ 8+12a^{-\frac{1}{3}}} & \\
-\ 6-12a^{-\frac{1}{3}}+18a^{-\frac{2}{3}} & \\
\underline{-\ 6-12a^{-\frac{1}{3}}+18a^{-\frac{2}{3}}} &
\end{array}$$

Note. It is important to arrange the dividend, divisor, and each remainder in the same order of powers of some common letter.

Divide the following:

11. a^2+b^2 by $a^{\frac{2}{3}}+b^{\frac{2}{3}}$. **12.** $a^{-1}-1$ by $a^{-\frac{1}{4}}-1$.

13. x^4-2+x^{-4} by x^2+2+x^{-2}.

14. $a-4a^{\frac{3}{4}}+6a^{\frac{1}{2}}-4a^{\frac{1}{4}}+1$ by $a^{\frac{1}{2}}-2a^{\frac{1}{4}}+1$.

15. $x^{\frac{3}{2}}-3xy^{\frac{1}{3}}+3x^{\frac{1}{2}}y^{\frac{2}{3}}-y$ by $x^{\frac{1}{2}}-y^{\frac{1}{3}}$.

16. $m^{-5}-3m^{-3}-4m^{-1}$ by $m^{-3}+2m^{-2}$.

17. $9x^2y^2+5+x^{-2}y^{-2}$ by $3x^{-1}-x^{-2}y^{-1}+x^{-3}y^{-2}$.

18. $a^{-1}m-5am^{-1}+4a^3m^{-3}$ by $a^{-3}m^2-a^{-2}m-2a^{-1}$.

19. $ab^{-\frac{1}{3}}-10b^{\frac{1}{3}}+9a^{-1}b$ by $a^{\frac{1}{2}}+2b^{\frac{1}{3}}-3a^{-\frac{1}{2}}b^{\frac{2}{3}}$.

20. $m^{\frac{4}{3}}-2x^{\frac{3}{2}}+m^{-\frac{4}{3}}x^3$ by $m^{-\frac{2}{3}}x^{\frac{3}{4}}-2m^{-\frac{4}{3}}x^{\frac{3}{2}}+m^{-2}x^{\frac{9}{4}}$.

219. *To prove* $(a^m)^n=a^{mn}$ *for all values of* m *and* n.

We will consider three cases, in each of which m may have any value, positive or negative, integral or fractional.

I. Let n be a positive integer.

The proof of (2), § 209, holds if n is a positive integer, whatever the value of m.

II. Let $n=\frac{p}{q}$, where p and q are positive integers.

Then, by the definition of § 211,

$$(a^m)^{\frac{p}{q}}=\sqrt[q]{(a^m)^p}=\sqrt[q]{a^{mp}}\text{ (§ 219, I.)}=a^{\frac{mp}{q}}.$$

III. Let $n=-s$, where s is a positive number.

Then, by the definition of § 214,

$$(a^m)^{-s}=\frac{1}{(a^m)^s}=\frac{1}{a^{ms}}\text{(§ 219, I. or II.)}=a^{-ms}.$$

Therefore, the equation holds for all values of m and n.

For example, $(a^2)^{-5} = a^{2\times-5} = a^{-10}$;

$$(a^{-3})^{-\frac{1}{3}} = a^{-3\times-\frac{1}{3}} = a;$$

$$(\sqrt{a})^{\frac{2}{3}} = (a^{\frac{1}{2}})^{\frac{2}{3}} = a^{\frac{1}{2}\times\frac{2}{3}} = a^{\frac{1}{3}}; \text{ etc.}$$

EXAMPLES.

$-\frac{6}{3}\times$

220. Find the values of the following:

1. $(a^3)^{-2}$.
2. $(a^{-5})^4$.
3. $(x^7)^{\frac{1}{7}}$.
4. $(m^{-\frac{1}{5}})^{\frac{5}{6}}$.
5. $(b^{\frac{4}{5}})^{-5}$.
6. $(a^{-1})^{-\frac{4}{7}}$.
7. $(x^{-\frac{3}{2}})^2$.
8. $(\sqrt[5]{x^6})^{\frac{1}{2}}$.
9. $(a^{-4})^{\frac{7}{8}}$.
10. $(\sqrt[4]{m})^6$.
11. $\left(\frac{1}{a}\right)^{-6}$.
12. $(\sqrt[3]{x^2})^{-\frac{3}{4}}$.
13. $\left(\frac{1}{\sqrt[6]{c^5}}\right)^{-3}$.
14. $(a^{-\frac{3n}{4}})^{\frac{2}{9n}}$.
15. $[(\sqrt{m^3})^{-\frac{1}{2}}]^{\frac{1}{3}}$.
16. $(x^{\frac{m}{n}-1})^{\frac{m}{m-n}}$.

221. The value of a numerical quantity affected with a fractional exponent may be found by first, if possible, extracting the root indicated by the denominator, and then raising the result to the power indicated by the numerator.

1. Find the value of $(-8)^{\frac{2}{3}}$.

We have,

$$(-8)^{\frac{2}{3}} = [(-8)^{\frac{1}{3}}]^2 = (\sqrt[3]{-8})^2 = (-2)^2 = 4, \textit{ Ans.}$$

EXAMPLES.

Find the values of the following:

2. $25^{\frac{3}{2}}$.
3. $9^{\frac{5}{2}}$.
4. $8^{\frac{5}{3}}$.
5. $81^{\frac{3}{4}}$.
6. $49^{-\frac{1}{2}}$.
7. $(-27)^{-\frac{1}{3}}$.
8. $4^{\frac{7}{2}}$.
9. $343^{\frac{2}{3}}$.
10. $16^{-\frac{5}{4}}$.
11. $(-32)^{\frac{7}{5}}$.
12. $64^{\frac{5}{6}}$.
13. $(-125)^{\frac{4}{3}}$.
14. $32^{-\frac{6}{5}}$.
15. $(-64)^{\frac{5}{3}}$.
16. $(-243)^{\frac{4}{5}}$.
17. $(-128)^{-\frac{3}{7}}$.

222. *To prove* $(ab)^n = a^n b^n$ *for any value of* n.

I. Let n be a positive integer.

$$\text{Then, } (ab)^n = ab \times ab \times ab \times \cdots \text{ to } n \text{ factors}$$
$$= (a \times a \times \cdots \text{ to } n \text{ factors})(b \times b \times \cdots \text{ to } n \text{ factors})$$
$$= a^n b^n.$$

II. Let $n = \frac{p}{q}$, where p and q are positive integers.

Then by § 219, $[(ab)^{\frac{p}{q}}]^q = (ab)^p = a^p b^p$, by § 222, I.

And by § 222, I., $[a^{\frac{p}{q}} b^{\frac{p}{q}}]^q = (a^{\frac{p}{q}})^q (b^{\frac{p}{q}})^q = a^p b^p$.

Therefore, $[(ab)^{\frac{p}{q}}]^q = [a^{\frac{p}{q}} b^{\frac{p}{q}}]^q$.

Taking the qth root of both members, we have

$$(ab)^{\frac{p}{q}} = a^{\frac{p}{q}} b^{\frac{p}{q}}.$$

III. Let $n = -s$, where s is any positive number.

Then, $(ab)^{-s} = \frac{1}{(ab)^s} = \frac{1}{a^s b^s}$ (§ 222, I. or II.) $= a^{-s} b^{-s}$.

MISCELLANEOUS EXAMPLES.

223. Square the following by the rule of § 78 or § 79:

1. $2a^{\frac{3}{4}} + 3b^{\frac{1}{2}}$. **2.** $5x^{-3}y^{2} - 2x^{3}y^{-2}$. **3.** $3a^{\frac{2}{3}}x^{-\frac{1}{4}} - 4a^{-\frac{1}{6}}y^{\frac{4}{5}}$.

Extract the square roots of the following:

4. $a^{-4}b^{\frac{2}{3}}$. **5.** $25mn^{-\frac{2}{3}}x^{\frac{1}{3}}$. **6.** $\frac{a^{-\frac{4}{5}}b^{-2}}{c^{2}d^{\frac{1}{2}}}$. **7.** $\frac{a^{5}x^{-\frac{5}{4}}}{9b^{\frac{4}{3}}y^{-7}}$.

8. $4a^{\frac{1}{2}} + 4a^{\frac{1}{4}} - 19 - 10a^{-\frac{1}{4}} + 25a^{-\frac{1}{2}}$.

9. $9x^{-\frac{8}{3}} - 12x^{-2} + 10x^{-\frac{4}{3}} - 4x^{-\frac{2}{3}} + 1$.

10. $a^{3}b^{-4} - 8a^{\frac{5}{2}}b^{-5} + 10a^{2}b^{-6} + 24a^{\frac{3}{2}}b^{-7} + 9ab^{-8}$.

Extract the cube roots of the following:

11. a^4b^{-3}. **12.** $xy^{\frac{3}{2}}z^{-\frac{1}{3}}$. **13.** $-27m^{\frac{1}{2}}n^{-\frac{5}{4}}$. **14.** $\dfrac{a^{-2}x^{\frac{4}{3}}}{8y^{-\frac{3}{4}}}$.

15. $8a^4-12a^{\frac{8}{3}}b^{-\frac{1}{6}}+6a^{\frac{4}{3}}b^{-\frac{1}{3}}-b^{-\frac{1}{2}}$.

16. $x^{-\frac{3}{2}}+6x^{-\frac{1}{2}}+3x^{\frac{1}{2}}-28x^{\frac{3}{2}}-9x^{\frac{5}{2}}+54x^{\frac{7}{2}}-27x^{\frac{9}{2}}$.

Simplify the following:

17. $\dfrac{a^{2m-3n}\times a^{-5m-n}}{a^{3m-4n}}$.

18. $\dfrac{a^n\times\cancel{(a^{n-1})}^n}{a^{n+1}\times \cancel{a^{n-1}}}$.

19. $\left(x^{\frac{1}{n-1}}\times x^{\frac{1}{n+1}}\right)^{\frac{1}{n}}$.

20. $\left[(x^m)^{m-\frac{1}{m}}\right]^{\frac{1}{m+1}}$.

21. $\left(\dfrac{x^{m+n}}{x^n}\right)^m\times\left(\dfrac{x^n}{x^{m+n}}\right)^{m-n}$.

22. $\left(a^{\frac{x}{x+y}}\div a^{\frac{x-y}{x}}\right)^{\frac{x+y}{y}}$.

23. $(2^{n+4}-2\times 2^n)\times(2^{-2}\times 2^{-n-2})$.

24. $\left(x^{\frac{n}{m-n}}\right)^{m^2-n^2}\div\left(\dfrac{x^n}{x^m}\right)^n$.

25. $\dfrac{(x^{\frac{1}{2}}-a^{\frac{1}{2}})^2+(1+x^{\frac{1}{2}}a^{\frac{1}{2}})^2}{1+a^{\frac{1}{2}}x^{\frac{1}{2}}(a^{-\frac{1}{2}}x^{\frac{1}{2}}-a^{\frac{1}{2}}x^{-\frac{1}{2}}-a^{\frac{1}{2}}x^{\frac{1}{2}})}$.

26. $\dfrac{x+y}{x^{\frac{1}{3}}-y^{\frac{1}{3}}}+\dfrac{x-y}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}$.

27. $\dfrac{a^{\frac{3n}{2}}-a^{-\frac{3n}{2}}}{a^{\frac{n}{2}}-a^{-\frac{n}{2}}}$.

28. $\dfrac{x^{-\frac{3}{2}}-y^{-\frac{3}{2}}}{x^{-\frac{1}{2}}+y^{-\frac{1}{2}}}-\dfrac{x^{-\frac{3}{2}}+y^{-\frac{3}{2}}}{x^{-\frac{1}{2}}-y^{-\frac{1}{2}}}$.

29. $\dfrac{x^{\frac{3n}{2}}-1}{x^{\frac{n}{2}}-1}-\dfrac{x^n-1}{x^{\frac{n}{2}}+1}$.

30. $\dfrac{x^{-1}+y^{-1}}{x^{-1}-y^{-1}}\times\dfrac{x^{-2}-y^{-2}}{x^{-2}+y^{-2}}-1$.

31. $\dfrac{a^{\frac{1}{3}}+2b^{\frac{1}{3}}}{a^{\frac{1}{3}}-2b^{\frac{1}{3}}}+\dfrac{a^{\frac{2}{3}}-2a^{\frac{1}{3}}b^{\frac{1}{3}}+4b^{\frac{2}{3}}}{a^{\frac{2}{3}}+2a^{\frac{1}{3}}b^{\frac{1}{3}}+4b^{\frac{2}{3}}}$.

XXI. RADICALS.

224. A **Radical** is a root of an expression, indicated by a radical sign; as $\sqrt{a}$, or $\sqrt[3]{x+1}$.

If the indicated root can be exactly obtained, the radical is called a *rational* quantity; if it cannot be exactly obtained, it is called an *irrational* quantity, or *surd.*

225. The *degree* of a radical is denoted by the index of the radical sign; thus, $\sqrt[3]{x+1}$ is of the *third* degree.

226. Most problems in radicals depend for their solution on the following principle:

For any value of n, $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} \times b^{\frac{1}{n}}$. (§ 222)

That is, $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$.

REDUCTION OF A RADICAL TO ITS SIMPLEST FORM.

227. A radical is said to be in its *simplest form* when the expression under the radical sign is integral, is not a perfect power of the degree denoted by any factor of the index of the radical, and has no factor which is a perfect power of the same degree as the radical.

228. Case I. *When the expression under the radical sign is a perfect power of the degree denoted by a factor of the index.*

1. Reduce $\sqrt[6]{8}$ to its simplest form.

We have, $\sqrt[6]{8} = \sqrt[6]{2^3} = 2^{\frac{3}{6}}$ (§ 211) $= 2^{\frac{1}{2}} = \sqrt{2}$, *Ans.*

EXAMPLES.

Reduce the following to their simplest forms:

2. $\sqrt[4]{36}$. 3. $\sqrt[6]{49}$. 4. $\sqrt[8]{25}$. 5. $\sqrt[9]{64}$.

6. $\sqrt[10]{9}$. 9. $\sqrt[15]{216}$. 12. $\sqrt[4]{121\,a^2b^4}$. 15. $\sqrt[8]{81\,m^8n^{12}}$.
7. $\sqrt[12]{16}$. 10. $\sqrt[12]{100}$. 13. $\sqrt[6]{125\,x^3y^9}$. 16. $\sqrt[12]{8\,x^9m^6}$.
8. $\sqrt[14]{4}$. 11. $\sqrt[15]{243}$. 14. $\sqrt[10]{32\,a^{15}m^5}$. 17. $\sqrt[9]{27\,a^6x^3}$.

229. Case II. *When the expression under the radical sign is integral, and has a factor which is a perfect power of the same degree as the radical.*

1. Reduce $\sqrt[3]{54}$ to its simplest form.

We have, $\sqrt[3]{54} = \sqrt[3]{27 \times 2} = \sqrt[3]{27} \times \sqrt[3]{2}$ (§ 226) $= 3\sqrt[3]{2}$, *Ans.*

2. Reduce $\sqrt{3\,a^3b - 12\,a^2b^2 + 12\,ab^3}$ to its simplest form.

$$\sqrt{3\,a^3b - 12\,a^2b^2 + 12\,ab^3} = \sqrt{(a^2 - 4\,ab + 4\,b^2)3\,ab}$$
$$= \sqrt{a^2 - 4\,ab + 4\,b^2}\sqrt{3\,ab} = (a - 2\,b)\sqrt{3\,ab}, \textit{ Ans.}$$

From the above examples, we derive the following rule:

Resolve the expression under the radical sign into two factors, the second of which contains no factor which is a perfect power of the same degree as the radical.

Extract the required root of the first factor, and prefix the result to the indicated root of the second.

EXAMPLES.

Reduce the following to their simplest forms:

3. $\sqrt{28}$. 7. $3\sqrt{98}$. 11. $\sqrt[3]{375}$. 15. $\sqrt{192\,m^5n^7}$.
4. $\sqrt{99}$. 8. $\sqrt{150}$. 12. $\sqrt[4]{162}$. 16. $\sqrt[3]{128\,xy^3z^5}$.
5. $\sqrt{80}$. 9. $5\sqrt[3]{108}$. 13. $\sqrt[6]{128}$. 17. $\sqrt[4]{64\,a^6b^9}$.
6. $\sqrt[3]{40}$. 10. $\sqrt{243}$. 14. $\sqrt{242\,a^3x^2}$. 18. $\sqrt[5]{96\,a^5b^9c^7}$.
19. $\sqrt{108\,a^3b^6 + 72\,a^4b^5}$. 21. $\sqrt{(a^2 - 4\,b^2)(a - 2\,b)}$.
20. $\sqrt[3]{135\,x^5y^5 - 108\,x^3y^7}$. 22. $\sqrt{5\,x^3 + 30\,x^2 + 45\,x}$.
23. $\sqrt{27\,a^3b - 36\,a^2b^2 + 12\,ab^3}$.
24. $\sqrt{(x^2 - x - 6)(x^2 + 2\,x - 15)}$.

If the expression under the radical sign has a numerical factor which cannot be readily factored by inspection, it is convenient to resolve it into its prime factors.

25. Reduce $\sqrt[3]{1944}$ to its simplest form.

$$\sqrt[3]{1944} = \sqrt[3]{2^3 \times 3^5} = \sqrt[3]{2^3 \times 3^3} \times \sqrt[3]{3^2} = 2 \times 3 \times \sqrt[3]{9} = 6\sqrt[3]{9}, \textit{ Ans.}$$

26. Reduce $\sqrt{125 \times 147}$ to its simplest form.

$$\sqrt{125 \times 147} = \sqrt{5^3 \times 3 \times 7^2} = \sqrt{5^2 \times 7^2} \times \sqrt{3 \times 5} = 5 \times 7 \times \sqrt{15} = 35\sqrt{15}, \textit{ Ans.}$$

Reduce the following to their simplest forms:

27. $\sqrt{864}$.

28. $\sqrt{2625}$.

29. $\sqrt{3528}$.

30. $\sqrt{125 \times 135}$.

31. $\sqrt{98 \times 336}$.

32. $\sqrt[3]{1125}$.

33. $\sqrt[3]{4116}$.

34. $\sqrt[3]{196 \times 392}$.

35. $\sqrt[3]{40 \times 45 \times 48}$.

36. $\sqrt{75a^2 \times 105ab \times 189b^2}$.

230. CASE III. *When the expression under the radical sign is a fraction.*

In this case, the radical may be reduced to its simplest form by *multiplying both terms of the fraction by such an expression as will make the denominator a perfect power of the same degree as the radical, and then proceeding as in* § 229.

1. Reduce $\sqrt{\frac{9}{8a^3}}$ to its simplest form.

Multiplying both terms of the fraction by $2a$, we have

$$\sqrt{\frac{9}{8a^3}} = \sqrt{\frac{9 \times 2a}{16a^4}} = \sqrt{\frac{9}{16a^4} \times 2a} = \sqrt{\frac{9}{16a^4}} \times \sqrt{2a} = \frac{3}{4a^2}\sqrt{2a}, \textit{ Ans.}$$

EXAMPLES.

Reduce the following to their simplest forms:

2. $\sqrt{\frac{2}{3}}$.

3. $\sqrt{\frac{4}{5}}$.

4. $\sqrt{\frac{5}{12}}$.

5. $\sqrt{\frac{9}{8}}$.

6. $\sqrt{\frac{13}{20}}$.

7. $\sqrt{\frac{17}{32}}$.

8. $\sqrt[3]{\frac{3}{2}}$.

9. $\sqrt[3]{\frac{1}{4}}$.

10. $\sqrt[3]{\frac{7}{9}}$.

11. $\sqrt[3]{\frac{8}{25}}$.

12. $\sqrt[4]{\frac{2}{3}}$.

13. $\sqrt[4]{\frac{5}{8}}$.

14. $\sqrt[5]{\frac{1}{3}}$.

15. $\sqrt[5]{\frac{9}{16}}$.

16. $\sqrt{\frac{7}{6a}}$.

17. $\sqrt{\frac{3}{10x^3}}$.

18. $\sqrt{\frac{11a^3b^4}{18c^2d^5}}$.

19. $\sqrt{\frac{16xy^3}{27z^5}}$.

20. $\sqrt[3]{\frac{2}{7a}}$.

21. $\sqrt[3]{\frac{5x^2}{16y^2}}$.

22. $\sqrt{\frac{a+b}{a-b}}$.

23. $\frac{x}{x^2-4}\sqrt{\frac{2x^3-8x+8}{x}}$.

231. To Introduce the Coefficient of a Radical under the Radical Sign.

The coefficient of a radical may be introduced under the radical sign by raising it to the power denoted by the index.

1. Introduce the coefficient of $2a\sqrt[3]{3x^2}$ under the radical sign.

$2a\sqrt[3]{3x^2}=\sqrt[3]{8a^3}\sqrt[3]{3x^2}=\sqrt[3]{8a^3\times 3x^2}$ (§ 226) $=\sqrt[3]{24a^3x^2}$, *Ans.*

Note. A rational quantity may be expressed in the form of a radical by raising it to the power denoted by the index, and writing the result under the corresponding radical sign.

EXAMPLES.

Introduce under the radical signs the coefficients of:

2. $5\sqrt{2}$.

3. $8\sqrt{3}$.

4. $4\sqrt[3]{6}$.

5. $5\sqrt[3]{4}$.

6. $2\sqrt[4]{5}$.

7. $3\sqrt[5]{2}$.

8. $4a\sqrt{8a}$.

9. $7x^2\sqrt{6x^3}$.

10. $3ab\sqrt[3]{5a^2}$.

11. $x^3y^2\sqrt[3]{x^2y^4}$.

12. $3m^2\sqrt[4]{2m}$.

13. $2a\sqrt[5]{7a^3}$.

14. $(1+a)\sqrt{\frac{1-a}{1+a}}$.

15. $(x-1)\sqrt{\frac{2}{x-1}+1}$.

16. $\frac{a-b}{a+b}\sqrt{\frac{a+b}{a-b}}$.

17. $\frac{x^2-1}{x^2+1}\sqrt{1-\frac{2x}{(x+1)^2}}$.

232. *Similar radicals* are radicals which do not differ at all, or differ only in their coefficients; as $2\sqrt[3]{ax^2}$ and $3\sqrt[3]{ax^2}$.

ADDITION AND SUBTRACTION OF RADICALS.

233. To add or subtract *similar radicals* (§ 232), add or subtract their coefficients, and prefix the result to their common radical part.

1. Required the sum of $\sqrt{20}$ and $\sqrt{45}$.

Reducing each radical to its simplest form (§ 229), we have

$$\sqrt{20}+\sqrt{45}=\sqrt{4\times5}+\sqrt{9\times5}=2\sqrt{5}+3\sqrt{5}=5\sqrt{5},\ \textit{Ans.}$$

2. Simplify $\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}-\sqrt{\frac{9}{8}}$.

$$\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}-\sqrt{\frac{9}{8}}=\sqrt{\frac{1}{4}\times2}+\sqrt{\frac{1}{9}\times6}-\sqrt{\frac{9}{16}\times2}$$

$$=\tfrac{1}{2}\sqrt{2}+\tfrac{1}{3}\sqrt{6}-\tfrac{3}{4}\sqrt{2}=\tfrac{1}{3}\sqrt{6}-\tfrac{1}{4}\sqrt{2},\ \textit{Ans.}$$

RULE.

Reduce each radical to its simplest form.

Unite the similar radicals, and indicate the addition or subtraction of those which are not similar.

EXAMPLES.

Simplify the following:

3. $\sqrt{75}+\sqrt{12}$. 5. $\sqrt{80}-\sqrt{180}$. 7. $\sqrt[3]{192}-\sqrt[3]{3}$.

4. $\sqrt{98}-\sqrt{18}$. 6. $\sqrt[3]{54}+\sqrt[3]{16}$. 8. $\sqrt[4]{32}-\sqrt[4]{162}$.

9. $\sqrt{27}+\sqrt{108}-\sqrt{48}$. 10. $\sqrt{175}-\sqrt{112}-\sqrt{44}$.

11. $\sqrt{\frac{9}{2}}+\sqrt{\frac{25}{8}}$. 12. $\sqrt{\frac{2}{27}}+\sqrt{\frac{8}{3}}$. 13. $\sqrt[3]{\frac{2}{9}}+\sqrt[3]{\frac{1}{36}}$.

14. $\sqrt{5}+\sqrt{245}-\sqrt{320}$. 16. $\sqrt[3]{5}-\sqrt[3]{320}+\sqrt[3]{72}$.

15. $\sqrt{\frac{8}{5}}+\sqrt{\frac{9}{10}}-\sqrt{\frac{5}{8}}$. 17. $\sqrt{\frac{16}{15}}-\sqrt{\frac{20}{3}}+\sqrt{\frac{3}{5}}$.

18. $b^2\sqrt{8a^5b} + ab\sqrt{50a^3b^3} - a^2\sqrt{128ab^5}$.

19. $m^2\sqrt[3]{32m^2} + m\sqrt[3]{108m^5} + \sqrt[3]{500m^8}$.

20. $\sqrt{50a^4 - 75a^2x} - \sqrt{32a^2x^4 - 48x^5}$.

21. $\sqrt{\frac{7}{2}} + \sqrt{\frac{18}{7}} - \sqrt{\frac{25}{14}}$.

22. $\sqrt[3]{\frac{9}{2}} + \sqrt[3]{\frac{4}{3}} + \sqrt[3]{\frac{1}{6}}$.

23. $\sqrt[3]{81} - \sqrt[3]{24} - \sqrt[3]{48} + \sqrt[3]{375}$.

24. $\sqrt[4]{243} - \sqrt[4]{48} - \sqrt[4]{768}$.

25. $\sqrt{32} - \sqrt{72} + \sqrt{125} + \sqrt{162} - \sqrt{500}$.

26. $x^2\sqrt{150x} + \sqrt{96x^3} - \sqrt{54x^5} - x\sqrt{24x^3}$.

27. $\sqrt{63a^2b} + b\sqrt{160ab^3} - \sqrt{40ab^5} - a\sqrt{252b}$.

28. $\sqrt{\frac{3}{2}} - \sqrt{\frac{1}{6}} - \sqrt{\frac{32}{3}} + \sqrt{\frac{1}{27}}$.

29. $\sqrt{\frac{15}{32}} + \sqrt{\frac{1}{10}} + \sqrt{\frac{6}{5}} - \sqrt{\frac{5}{2}}$.

30. $\sqrt{80x^3 + 40x^2 + 5x} + \sqrt{45x^3 - 60x^2 + 20x}$.

31. $2\sqrt{12x^2 + 60xy + 75y^2} - \sqrt{48x^2 - 72xy + 27y^2}$.

32. $\sqrt{\frac{a+b}{a-b}} - \sqrt{\frac{a-b}{a+b}} + \frac{2a}{a^2-b^2}\sqrt{a^2-b^2}$.

TO REDUCE RADICALS OF DIFFERENT DEGREES TO EQUIVALENT RADICALS OF THE SAME DEGREE.

234. 1. Reduce $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$ to equivalent radicals of the same degree.

By § 211,
$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{6}{12}} = \sqrt[12]{2^6} = \sqrt[12]{64};$$
$$\sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{4}{12}} = \sqrt[12]{3^4} = \sqrt[12]{81};$$
$$\sqrt[4]{5} = 5^{\frac{1}{4}} = 5^{\frac{3}{12}} = \sqrt[12]{5^3} = \sqrt[12]{125}.$$

We then have the following rule:

Express the radicals with fractional exponents, and reduce these exponents to a common denominator.

Note. The relative magnitude of radicals may be determined by reducing them, if necessary, to radicals of the same degree.

Thus, in Ex. 1, $\sqrt[12]{125}$ is greater than $\sqrt[12]{81}$, and $\sqrt[12]{81}$ than $\sqrt[12]{64}$.

Hence, $\sqrt[4]{5}$ is greater than $\sqrt[3]{3}$, and $\sqrt[3]{3}$ than $\sqrt{2}$.

EXAMPLES.

Reduce to equivalent radicals of the same degree:

2. $\sqrt{3}$ and $\sqrt[3]{5}$.

3. $\sqrt{2}$ and $\sqrt[5]{3}$.

4. $\sqrt[3]{a^2b}$ and $\sqrt[5]{a^4b^3}$.

5. $\sqrt{2}$ and $\sqrt[7]{12}$.

6. $\sqrt[3]{4}$ and $\sqrt[4]{6}$.

7. $\sqrt{xy}$, $\sqrt[4]{yz}$, and $\sqrt[5]{zx}$.

8. $\sqrt[3]{3a}$, $\sqrt[4]{2b}$, and $\sqrt[6]{6c}$.

9. $\sqrt[3]{2}$, $\sqrt[6]{8}$, and $\sqrt[9]{13}$.

10. $\sqrt[4]{1-x}$ and $\sqrt[6]{1+x}$.

11. $\sqrt[3]{a+b}$ and $\sqrt[6]{a-b}$.

12. Which is the greater, $\sqrt[3]{2}$ or $\sqrt[4]{3}$?

13. Which is the greater, $\sqrt[3]{11}$ or $\sqrt{5}$?

14. Which is the greater, $\sqrt[5]{10}$ or $\sqrt[3]{4}$?

15. Arrange in order of magnitude, $\sqrt[3]{14}$, $\sqrt{6}$, and $\sqrt[6]{175}$.

16. Arrange in order of magnitude, $\sqrt{3}$, $\sqrt[5]{15}$, and $\sqrt[10]{253}$.

17. Arrange in order of magnitude, $\sqrt{3}$, $\sqrt[3]{5}$, and $\sqrt[4]{7}$.

MULTIPLICATION OF RADICALS.

235. 1. Multiply $\sqrt{6}$ by $\sqrt{15}$.

$$\sqrt{6}\times\sqrt{15}=\sqrt{6\times 15}\ (\S\ 226)=\sqrt{2\times 3\times 3\times 5}$$
$$=\sqrt{3^2}\times\sqrt{2\times 5}=3\sqrt{10},\ \textit{Ans.}$$

2. Multiply $\sqrt{2a}$ by $\sqrt[3]{4a^2}$.

Reducing to equivalent radicals of the same degree,

$$\sqrt{2a} \times \sqrt[3]{4a^2} = (2a)^{\frac{1}{2}} \times (4a^2)^{\frac{1}{3}} = (2a)^{\frac{3}{6}} \times (4a^2)^{\frac{2}{6}} = \sqrt[6]{(2a)^3} \times \sqrt[6]{(4a^2)^2}$$
$$= \sqrt[6]{2^3a^3 \times 2^4a^4} = \sqrt[6]{2^6a^6} \times \sqrt[6]{2a} = 2a\sqrt[6]{2a}, \text{ *Ans.*}$$

3. Multiply $\sqrt{20}$ by $\sqrt[6]{5}$.

We have, $\sqrt{20} = 20^{\frac{1}{2}} = 20^{\frac{3}{6}} = \sqrt[6]{20^3} = \sqrt[6]{(2^2 \times 5)^3} = \sqrt[6]{2^6 \times 5^3}$.

Whence, $\sqrt{20} \times \sqrt[6]{5} = \sqrt[6]{2^6 \times 5^3 \times 5} = \sqrt[6]{2^6} \times \sqrt[6]{5^4} = 2 \times 5^{\frac{4}{6}} = 2 \times 5^{\frac{2}{3}}$
$$= 2 \times \sqrt[3]{5^2} = 2\sqrt[3]{25}, \text{ *Ans.*}$$

From the above examples, we have the following rule:

Reduce the radicals, if necessary, to equivalent radicals of the same degree.

Multiply together the expressions under the radical signs, and write the result under the common radical sign.

The result should be reduced to its simplest form.

EXAMPLES.

Multiply the following:

4. $\sqrt{3}$ and $\sqrt{48}$.
5. $\sqrt[3]{6a^2}$ and $\sqrt[3]{36a}$.
6. $\sqrt{14}$ and $\sqrt{18}$.
7. $\sqrt{15}$ and $\sqrt{50}$.
8. $\sqrt{44}$ and $\sqrt{275}$.
9. $\sqrt{30ab}$ and $\sqrt{70ac}$.
10. $\sqrt[3]{36}$ and $\sqrt[3]{48}$.
11. $\sqrt[3]{15}$ and $\sqrt[3]{63}$.
12. $\sqrt[3]{90}$ and $\sqrt[3]{132}$.
13. $\sqrt{\frac{21}{2}}$ and $\sqrt{\frac{35}{8}}$.
14. $\sqrt[4]{9}$ and $\sqrt[4]{135}$.
15. $\sqrt[5]{48}$ and $\sqrt[5]{56}$.
16. $\sqrt[3]{3x}$ and $\sqrt[4]{27x^3}$.
17. $\sqrt{6}$ and $\sqrt[3]{12}$.
18. $\sqrt[3]{5ax}$ and $\sqrt[9]{4bx}$.
19. $\sqrt{5}$ and $\sqrt[5]{125}$.
20. $\sqrt[3]{4ab^2}$ and $\sqrt[5]{8b^2c}$.
21. $\sqrt[3]{45}$ and $\sqrt[6]{9}$.
22. $\sqrt{12}$ and $\sqrt[10]{3}$.
23. $\sqrt[4]{54}$ and $\sqrt[8]{72}$.

24. $\sqrt{\frac{9}{8}}$ and $\sqrt[3]{\frac{4}{3}}$.

25. $\sqrt{\frac{5}{27}}$ and $\sqrt[4]{\frac{3}{5}}$.

26. $\sqrt{ab}$, $\sqrt[4]{bc}$, and $\sqrt[8]{ca}$.

27. $\sqrt{6}$, $\sqrt[5]{2}$, and $\sqrt[10]{8}$.

28. $\sqrt{2}$, $\sqrt[3]{4}$, and $\sqrt[4]{3}$.

29. $\sqrt[3]{12}$, $\sqrt[6]{\frac{1}{9}}$, and $\sqrt[9]{\frac{1}{8}}$.

30. Multiply $2\sqrt{3}+3\sqrt{2}$ by $3\sqrt{3}-\sqrt{2}$.

$$\begin{array}{l}
2\sqrt{3}+3\sqrt{2} \\
\underline{3\sqrt{3}-\ \sqrt{2}} \\
18+9\sqrt{6} \\
\underline{\qquad -2\sqrt{6}-6} \\
18+7\sqrt{6}-6=12+7\sqrt{6},\ \textit{Ans.}
\end{array}$$

Note. It should be remembered that to multiply a radical of the second degree by itself simply removes the radical sign; thus, $\sqrt{3}\times\sqrt{3}=3$.

31. Multiply $3\sqrt{x^2+1}+4x$ by $2\sqrt{x^2+1}-x$.

$$\begin{array}{l}
3\sqrt{x^2+1}+4x \\
\underline{2\sqrt{x^2+1}-\ x} \\
6(x^2+1)+8x\sqrt{x^2+1} \\
\underline{\qquad\qquad -3x\sqrt{x^2+1}-4x^2} \\
6x^2+6\ \ +5x\sqrt{x^2+1}-4x^2=2x^2+6+5x\sqrt{x^2+1},\ \textit{Ans.}
\end{array}$$

Multiply the following:

32. $5-2\sqrt{3}$ and $4+3\sqrt{3}$.

33. $2\sqrt{x}+3\sqrt{2}$ and $6\sqrt{x}-\sqrt{2}$.

34. $7\sqrt{2}-4\sqrt{5}$ and $3\sqrt{2}-8\sqrt{5}$.

35. $6\sqrt{a}+11\sqrt{b}$ and $9\sqrt{a}-5\sqrt{b}$.

36. $5\sqrt[3]{4}+3\sqrt[3]{5}$ and $6\sqrt[3]{2}+7\sqrt[3]{25}$.

37. $\sqrt{a}+2\sqrt{b}-3\sqrt{c}$ and $\sqrt{a}-2\sqrt{b}-3\sqrt{c}$.

38. $4\sqrt{x+1}-5\sqrt{x-1}$ and $3\sqrt{x+1}-2\sqrt{x-1}$.

39. $\sqrt{6}-\sqrt{3}-\sqrt{5}$ and $\sqrt{6}+\sqrt{3}+\sqrt{5}$.

40. $9\sqrt{\frac{2}{3}}-8\sqrt{\frac{9}{8}}$ and $3\sqrt{\frac{2}{3}}+10\sqrt{\frac{9}{8}}$.

41. $5\sqrt{3}+3\sqrt{5}+4\sqrt{7}$ and $5\sqrt{3}+3\sqrt{5}-4\sqrt{7}$.

42. $3\sqrt{2}-4\sqrt{5}-2\sqrt{7}$ and $6\sqrt{2}-8\sqrt{5}+4\sqrt{7}$.

43. $5\sqrt{8}+6\sqrt{12}-2\sqrt{20}$ and $7\sqrt{2}-3\sqrt{3}+4\sqrt{5}$.

44. $6\sqrt{\frac{3}{2}}-9\sqrt{\frac{5}{3}}+10\sqrt{\frac{2}{5}}$ and $2\sqrt{\frac{3}{2}}+3\sqrt{\frac{5}{3}}-5\sqrt{\frac{2}{5}}$.

$$\frac{36 \quad 3}{108}$$

Expand the following by the rules of §§ 78, 79, or 80:

45. $(3\sqrt{5}+4)^2$.

46. $(5-2\sqrt{3})^2$.

47. $(6\sqrt{2}-4\sqrt{6})^2$.

48. $(7\sqrt{10}+5\sqrt{7})^2$.

49. $(\sqrt{2a}+\sqrt{3a-4})^2$.

50. $(3\sqrt{x+y}-2\sqrt{x-y})^2$.

51. $(3\sqrt{2}+7)(3\sqrt{2}-7)$.

52. $(6\sqrt{3}+4\sqrt{5})(6\sqrt{3}-4\sqrt{5})$.

53. $(2\sqrt{x+1}+5\sqrt{x})(2\sqrt{x+1}-5\sqrt{x})$.

54. $(\sqrt{a+b}+\sqrt{a-b})(\sqrt{a+b}-\sqrt{a-b})$.

55. $(3\sqrt{2a-5}+4\sqrt{4a-3})(3\sqrt{2a-5}-4\sqrt{4a-3})$.

DIVISION OF RADICALS.

236. By § 226, $\sqrt[n]{ab}=\sqrt[n]{a}\times\sqrt[n]{b}$.

Whence, $$\frac{\sqrt[n]{ab}}{\sqrt[n]{a}}=\sqrt[n]{b}.$$

We then have the following rule:

Reduce the radicals, if necessary, to equivalent radicals of the same degree.

Divide the expression under the radical sign in the dividend by the expression under the radical sign in the divisor, and write the result under the common radical sign.

EXAMPLES.

1. Divide $\sqrt[3]{15}$ by $\sqrt{6}$.

Reducing to equivalent radicals of the same degree, we have

$$\frac{\sqrt[3]{15}}{\sqrt{6}}=\frac{15^{\frac{1}{3}}}{6^{\frac{1}{2}}}=\frac{15^{\frac{2}{6}}}{6^{\frac{3}{6}}}=\frac{\sqrt[6]{(3\times 5)^2}}{\sqrt[6]{(2\times 3)^3}}=\sqrt[6]{\frac{3^2\times 5^2}{2^3\times 3^3}}=\sqrt[6]{\frac{5^2}{2^3\times 3}}=\sqrt[6]{\frac{25}{24}},\ \textit{Ans.}$$

2. Divide $\sqrt{10}$ by $\sqrt[6]{40}$.

We have, $\sqrt{10}=10^{\frac{1}{2}}=10^{\frac{3}{6}}=\sqrt[6]{10^3}=\sqrt[6]{(2\times 5)^3}$.

Whence, $\dfrac{\sqrt{10}}{\sqrt[6]{40}}=\sqrt[6]{\dfrac{2^3\times 5^3}{2^3\times 5}}=\sqrt[6]{5^2}=5^{\frac{2}{6}}=5^{\frac{1}{3}}=\sqrt[3]{5}$, *Ans.*

Divide the following:

3. $\sqrt{84}$ by $\sqrt{7}$.
4. $\sqrt{12}$ by $\sqrt{15}$.
5. $\sqrt{56}$ by $\sqrt{32}$.
6. $\sqrt[3]{162}$ by $\sqrt[3]{2}$.
7. $\sqrt[3]{5}$ by $\sqrt[3]{135}$.
8. $\sqrt[3]{63}$ by $\sqrt[3]{35}$.
9. $\sqrt[3]{81}$ by $\sqrt[6]{9}$.
10. $\sqrt[5]{26a}$ by $\sqrt[5]{39a^3}$.
11. $\sqrt[4]{192}$ by $\sqrt[4]{3}$.
12. $\sqrt{2}$ by $\sqrt[10]{8}$.
13. $\sqrt[4]{5ab}$ by $\sqrt[3]{125b^2c}$.
14. $\sqrt{28x}$ by $\sqrt[4]{42x}$.
15. $\sqrt{\frac{21}{44}}$ by $\sqrt{\frac{28}{33}}$.
16. $\sqrt{2\frac{13}{16}}$ by $\sqrt{2\frac{1}{12}}$.
17. $\sqrt[3]{3m}$ by $\sqrt[4]{7m^3}$.
18. $\sqrt[3]{\frac{12}{25}}$ by $\sqrt[3]{\frac{9}{20}}$.
19. $\sqrt{6ax^2}$ by $\sqrt[3]{9a^2x^4}$.
20. $\sqrt{\frac{2}{5}}$ by $\sqrt[3]{\frac{4}{25}}$.
21. $\sqrt[5]{\frac{4}{9}}$ by $\sqrt{\frac{2}{3}}$.
22. $\sqrt[3]{14x^2y}$ by $\sqrt[5]{28xy^3}$.
23. $\sqrt[3]{20}$ by $\sqrt[9]{125}$.
24. $\sqrt[12]{\frac{27}{5}}$ by $\sqrt[4]{\frac{3}{5}}$.
25. $\sqrt[3]{12a^2}$ by $\sqrt{8a^3}$.
26. $\sqrt[4]{\frac{3}{8}}$ by $\sqrt[3]{\frac{9}{16}}$.

INVOLUTION OF RADICALS.

237. 1. Raise $\sqrt[6]{12}$ to the third power.

$$(\sqrt[6]{12})^3 = (12^{\frac{1}{6}})^3 = 12^{\frac{3}{6}} (\S\ 219) = 12^{\frac{1}{2}} = \sqrt{12} = 2\sqrt{3},\ \textit{Ans.}$$

2. Raise $\sqrt[5]{2}$ to the fourth power.

$$(\sqrt[5]{2})^4 = (2^{\frac{1}{5}})^4 = 2^{\frac{4}{5}} = \sqrt[5]{2^4} = \sqrt[5]{16},\ \textit{Ans.}$$

We then have the following rule:

If possible, divide the index of the radical by the exponent of the required power; otherwise, raise the expression under the radical sign to the required power.

EXAMPLES.

Find the values of the following:

3. $(\sqrt[5]{3})^4$.
4. $(\sqrt[4]{5})^2$.
5. $(\sqrt[3]{x+y})^2$.
6. $(3\sqrt[6]{16})^2$.
7. $(2\sqrt{a^3b})^5$.
8. $(\sqrt[9]{7a^4})^3$.
9. $(\sqrt[8]{50xy})^4$.
10. $(\sqrt[6]{3})^7$.
11. $(\sqrt[18]{128})^3$.
12. $(5m\sqrt[10]{96m^6})^2$.
13. $(\sqrt[7]{3a-2})^3$.
14. $(\sqrt[12]{48x^3y^5})^3$.

EVOLUTION OF RADICALS.

238. 1. Extract the cube root of $\sqrt[5]{27x^3}$.

$$\sqrt[3]{(\sqrt[5]{27x^3})} = (\sqrt[5]{(3x)^3})^{\frac{1}{3}} = [(3x)^{\frac{3}{5}}]^{\frac{1}{3}} = (3x)^{\frac{1}{5}} = \sqrt[5]{3x},\ \textit{Ans.}$$

2. Extract the fifth root of $\sqrt[3]{6}$.

$$\sqrt[5]{(\sqrt[3]{6})} = (6^{\frac{1}{3}})^{\frac{1}{5}} = 6^{\frac{1}{15}} = \sqrt[15]{6},\ \textit{Ans.}$$

We then have the following rule:

If possible, extract the required root of the expression under the radical sign; otherwise, multiply the index of the radical by the index of the required root.

Note. If the radical has a coefficient which is not a perfect power of the degree denoted by the index of the required root, it should be introduced under the radical sign (§ 231) before applying the rule.

Thus, $\sqrt[5]{4\sqrt{2}} = \sqrt[5]{\sqrt{32}} = \sqrt{2}$.

EXAMPLES.

Find the values of the following:

3. $\sqrt{(\sqrt[5]{49})}$.
4. $\sqrt[4]{(\sqrt{10})}$.
5. $\sqrt[5]{(\sqrt[6]{32a^5})}$.
6. $\sqrt[4]{(\sqrt[7]{81x^4y^8})}$.
7. $\sqrt{(16\sqrt[3]{9})}$.
8. $\sqrt[4]{(\sqrt[3]{25})}$.
9. $\sqrt{(\sqrt[4]{x^2-6x+9})}$.
10. $\sqrt[5]{(3\sqrt[4]{2xy^3})}$.
11. $\sqrt[6]{(2a\sqrt[3]{2a})}$.
12. $\sqrt[3]{(27x^3\sqrt[3]{5x})}$.
13. $\sqrt[6]{(\sqrt[4]{125})}$.
14. $\sqrt[3]{(3a^2\sqrt[4]{9a})}$.

TO REDUCE A FRACTION HAVING AN IRRATIONAL DENOMINATOR TO AN EQUIVALENT FRACTION WHOSE DENOMINATOR IS RATIONAL.

239. CASE I. *When the denominator is a monomial.*

The reduction may be effected by multiplying both terms of the fraction by a radical of the same degree as the denominator, having under its radical sign an expression which will make the denominator of the resulting fraction rational.

1. Reduce $\frac{5}{\sqrt[3]{3a^2}}$ to an equivalent fraction having a rational denominator.

Multiplying both terms of the fraction by $\sqrt[3]{9a}$, we have

$$\frac{5}{\sqrt[3]{3a^2}}=\frac{5\sqrt[3]{9a}}{\sqrt[3]{3a^2}\sqrt[3]{9a}}=\frac{5\sqrt[3]{9a}}{\sqrt[3]{27a^3}}=\frac{5\sqrt[3]{9a}}{3a},\ \textit{Ans.}$$

EXAMPLES.

Reduce each of the following to an equivalent fraction having a rational denominator:

2. $\frac{2}{\sqrt{6}}$.
3. $\frac{1}{\sqrt{7ab^3}}$.
4. $\frac{6}{\sqrt[3]{25}}$.
5. $\frac{4x}{\sqrt[3]{3x^2}}$.
6. $\frac{ab}{\sqrt[4]{4a^3b}}$.
7. $\frac{1}{\sqrt[4]{27}}$.
8. $\frac{5}{\sqrt[5]{8}}$.
9. $\frac{3}{\sqrt[6]{16x^3yz^5}}$.

240. Case II. *When the denominator is a binomial containing radicals of the second degree only.*

1. Reduce $\frac{5-\sqrt{3}}{5+\sqrt{3}}$ to an equivalent fraction having a rational denominator.

Multiplying both terms of the fraction by $5-\sqrt{3}$, we have

$$\frac{5-\sqrt{3}}{5+\sqrt{3}}=\frac{(5-\sqrt{3})^2}{(5+\sqrt{3})(5-\sqrt{3})}$$

$$=\frac{25-10\sqrt{3}+3}{25-3}(\S\S\ 79,\ 80)=\frac{28-10\sqrt{3}}{22}=\frac{14-5\sqrt{3}}{11},\ \textit{Ans.}$$

2. Reduce $\frac{3\sqrt{x}-2\sqrt{x-1}}{2\sqrt{x}-3\sqrt{x-1}}$ to an equivalent fraction having a rational denominator.

Multiplying both terms of the fraction by $2\sqrt{x}+3\sqrt{x-1}$,

$$\frac{3\sqrt{x}-2\sqrt{x-1}}{2\sqrt{x}-3\sqrt{x-1}}=\frac{(3\sqrt{x}-2\sqrt{x-1})(2\sqrt{x}+3\sqrt{x-1})}{(2\sqrt{x}-3\sqrt{x-1})(2\sqrt{x}+3\sqrt{x-1})}$$

$$=\frac{6x+5\sqrt{x}\sqrt{x-1}-6(x-1)}{4x-9(x-1)}=\frac{6+5\sqrt{x^2-x}}{9-5x},\ \textit{Ans.}$$

We then have the following rule:

Multiply both terms of the fraction by the denominator with the sign between its terms changed.

EXAMPLES.

Reduce each of the following to an equivalent fraction having a rational denominator:

3. $\frac{6}{3+\sqrt{5}}$.

4. $\frac{5}{2\sqrt{3}-4}$.

5. $\frac{\sqrt{a}+b}{\sqrt{a}-b}$.

6. $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$.

7. $\frac{5\sqrt{2}+\sqrt{6}}{3\sqrt{2}-\sqrt{6}}$.

8. $\frac{3\sqrt{5}+2\sqrt{2}}{3\sqrt{5}-2\sqrt{2}}$.

9. $\frac{3\sqrt{3}-2\sqrt{5}}{5\sqrt{3}+4\sqrt{5}}$.

10. $\frac{\sqrt{x-2}+1}{\sqrt{x-2}+2}$.

11. $\frac{\sqrt{a-b}+\sqrt{a}}{\sqrt{a-b}-\sqrt{a}}$.

12. $\dfrac{2x+\sqrt{1-4x^2}}{2x-\sqrt{1-4x^2}}$.

13. $\dfrac{a-\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}}$.

14. $\dfrac{\sqrt{1+a}-\sqrt{1-a}}{\sqrt{1+a}+\sqrt{1-a}}$.

15. $\dfrac{\sqrt{x-y}+\sqrt{x+y}}{\sqrt{x-y}-\sqrt{x+y}}$.

16. $\dfrac{\sqrt{a^2-x^2}-\sqrt{a^2+x^2}}{\sqrt{a^2-x^2}+\sqrt{a^2+x^2}}$.

17. $\dfrac{4\sqrt{x-1}+\sqrt{x+1}}{3\sqrt{x-1}-2\sqrt{x+1}}$.

241. The approximate value of a fraction whose denominator is irrational may be conveniently found by reducing it to an equivalent fraction with a rational denominator.

1. Find the approximate value of $\dfrac{1}{2-\sqrt{2}}$ to three places of decimals.

$$\frac{1}{2-\sqrt{2}}=\frac{2+\sqrt{2}}{(2-\sqrt{2})(2+\sqrt{2})}=\frac{2+\sqrt{2}}{4-2}=\frac{2+1.414\cdots}{2}=1.707\cdots, \textit{ Ans.}$$

EXAMPLES.

Find the approximate value of each of the following to three decimal places:

2. $\dfrac{3}{\sqrt{10}}$.

3. $\dfrac{1}{\sqrt{6}-2}$.

4. $\dfrac{4}{3+2\sqrt{5}}$.

5. $\dfrac{2}{\sqrt[3]{25}}$.

6. $\dfrac{5}{\sqrt{3}-2\sqrt{2}}$.

7. $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$.

8. $\dfrac{3\sqrt{2}-\sqrt{6}}{3\sqrt{2}+\sqrt{6}}$.

9. $\dfrac{3\sqrt{3}+2\sqrt{5}}{3\sqrt{3}-2\sqrt{5}}$.

10. $\dfrac{\sqrt{6}-4\sqrt{3}}{2\sqrt{6}+5\sqrt{3}}$.

PROPERTIES OF QUADRATIC SURDS.

242. A **Quadratic Surd** is the indicated square root of an imperfect square; as $\sqrt{3}$.

243. *A quadratic surd cannot be equal to the sum of a rational expression and a quadratic surd.*

For, if possible, let $\sqrt{a} = b + \sqrt{c}.$

Squaring both members, $a = b^2 + 2b\sqrt{c} + c.$

Transposing, $2b\sqrt{c} = a - b^2 - c.$

Dividing by $2b$, $\sqrt{c} = \frac{a - b^2 - c}{2b}.$

We then have a quadratic surd equal to a rational expression, which is impossible.

Hence, $\sqrt{a}$ cannot be equal to $b + \sqrt{c}.$

244. *If* $a + \sqrt{b} = c + \sqrt{d}$, *then* $a = c$, *and* $\sqrt{b} = \sqrt{d}.$

If a is not equal to c, let $a = c + x.$

Substituting this value in the given equation, we have

$$c + x + \sqrt{b} = c + \sqrt{d},$$

or,

$$x + \sqrt{b} = \sqrt{d}.$$

But this is impossible by § 243.

Hence, $a = c$, and therefore $\sqrt{b} = \sqrt{d}.$

245. *If* $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$, *then* $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}.$

Squaring both members of the given equation, we have

$$a + \sqrt{b} = x + 2\sqrt{xy} + y.$$

Whence by § 244, $a = x + y,$ (1)

and $\sqrt{b} = 2\sqrt{xy}.$ (2)

Subtracting (2) from (1),

$$a - \sqrt{b} = x - 2\sqrt{xy} + y.$$

Extracting the square root of both members,

$$\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}.$$

246. Square Root of a Binomial Surd.

The preceding principles may be used to find the square root of a binomial surd whose first term is rational.

Example. Required the square root of $13 - \sqrt{160}$.

Assume, $\sqrt{13 - \sqrt{160}} = \sqrt{x} - \sqrt{y}$. (1)

Then by § 245, $\sqrt{13 + \sqrt{160}} = \sqrt{x} + \sqrt{y}$. (2)

Multiplying (1) by (2), $\sqrt{169 - 160} = x - y$.

Or, $x - y = 3$. (3)

Squaring (1), $13 - \sqrt{160} = x - 2\sqrt{xy} + y$.

Whence by § 244, $x + y = 13$. (4)

Adding (3) and (4), $2x = 16$, or $x = 8$.

Subtracting (3) from (4), $2y = 10$, or $y = 5$.

Substituting in (1), $\sqrt{13 - \sqrt{160}} = \sqrt{8} - \sqrt{5} = 2\sqrt{2} - \sqrt{5}$, *Ans.*

247. Examples like that of § 246 may be solved by inspection by putting the given expression into the form of a perfect trinomial square (§ 96), as follows:

Reduce the surd term so that its coefficient may be 2.

Separate the rational term into two parts whose product shall be the expression under the radical sign of the surd term.

Extract the square root of each part, and connect the results by the sign of the surd term (§ 97).

1. Extract the square root of $8 + \sqrt{48}$.

We have, $\sqrt{8 + \sqrt{48}} = \sqrt{8 + 2\sqrt{12}}$.

We then separate 8 into two parts whose product is 12.
The parts are 6 and 2.

Whence, $\sqrt{8 + \sqrt{48}} = \sqrt{6 + 2\sqrt{12} + 2} = \sqrt{6} + \sqrt{2}$, *Ans.*

2. Extract the square root of $22 - 3\sqrt{32}$.

We have, $\sqrt{22 - 3\sqrt{32}} = \sqrt{22 - \sqrt{9 \times 8 \times 4}} = \sqrt{22 - 2\sqrt{72}}$.
We then separate 22 into two parts whose product is 72.
The parts are 18 and 4.

Whence, $\sqrt{22 - 3\sqrt{32}} = \sqrt{18} - \sqrt{4} = 3\sqrt{2} - 2$, *Ans.*

EXAMPLES.

Extract the square roots of the following:

3. $11+2\sqrt{28}$.

4. $17-2\sqrt{72}$.

5. $49+2\sqrt{48}$.

6. $19+4\sqrt{21}$.

7. $28-8\sqrt{6}$.

8. $30-2\sqrt{56}$.

9. $12-\sqrt{108}$.

10. $11+\sqrt{120}$.

11. $26+2\sqrt{160}$.

12. $20-6\sqrt{11}$.

13. $46-3\sqrt{20}$.

14. $35+10\sqrt{10}$.

15. $56+5\sqrt{48}$.

16. $35-12\sqrt{6}$.

17. $37-\sqrt{640}$.

18. $35-20\sqrt{3}$.

19. $85+5\sqrt{120}$.

20. $75-3\sqrt{96}$.

21. $2x+2\sqrt{x^2-1}$.

22. $a-2\sqrt{ab-b^2}$.

IMAGINARY NUMBERS.

248. An even root of a negative number is impossible; for no number when raised to an even power can produce a negative result (§ 186).

An **Imaginary Number** is an indicated even root of a negative number; as $\sqrt{-4}$, or $\sqrt[4]{-a^2}$.

In contradistinction, all other numbers, rational or irrational, are called *real* numbers.

249. Every imaginary square root can be expressed as the product of a real number by $\sqrt{-1}$.

Thus, $\sqrt{-a^2}=\sqrt{a^2\times(-1)}=\sqrt{a^2}\times\sqrt{-1}=a\sqrt{-1}$;

$$\sqrt{-5}=\sqrt{5\times(-1)}=\sqrt{5}\times\sqrt{-1};\text{ etc.}$$

250. To find the positive integral powers of $\sqrt{-1}$.

By § 189, $\sqrt{-1}$ signifies an expression whose square is equal to -1.

That is, $(\sqrt{-1})^2=-1$.

Then,

$$(\sqrt{-1})^3 = (\sqrt{-1})^2 \times \sqrt{-1} = (-1) \times \sqrt{-1} = -\sqrt{-1};$$
$$(\sqrt{-1})^4 = (\sqrt{-1})^2 \times (\sqrt{-1})^2 = (-1) \times (-1) = 1;$$
$$(\sqrt{-1})^5 = (\sqrt{-1})^4 \times \sqrt{-1} = 1 \times \sqrt{-1} = \sqrt{-1}; \text{ etc.}$$

Thus, the first four positive integral powers of $\sqrt{-1}$ are $\sqrt{-1}$, -1, $-\sqrt{-1}$, and 1; and for higher powers these terms recur in the same order.

251. Addition and Subtraction of Imaginary Numbers.

Imaginary numbers may be added or subtracted in the same manner as other radicals. (See § 233.)

1. Add $\sqrt{-4}$ and $\sqrt{-36}$.

$$\sqrt{-4} + \sqrt{-36} = 2\sqrt{-1} + 6\sqrt{-1} \text{ (§ 249)} = 8\sqrt{-1}, \textit{ Ans.}$$

EXAMPLES.

Simplify the following:

2. $\sqrt{-16}+\sqrt{-25}$.
3. $\sqrt{-3}+\sqrt{-27}$.
4. $\sqrt{-18}-\sqrt{-8}$.
5. $\sqrt{-a^2}-\sqrt{-(a-b)^2}$.
6. $\sqrt{-x^2}+\sqrt{-y^2}+\sqrt{-z^2}$.
7. $\sqrt{-1}+\sqrt{-49}-\sqrt{-64}$.
8. $\sqrt{-36}-\sqrt{-100}+\sqrt{-81}$.
9. $\sqrt{-a^2}-\sqrt{-4a^2}-\sqrt{-9a^2}$.
10. $\sqrt{-20}+\sqrt{-45}-\sqrt{-80}$.
11. $\sqrt{-1-2x-x^2}-\sqrt{-4x^2}$.

252. Multiplication of Imaginary Numbers.

The product of two or more imaginary square roots may be found by aid of the principles of §§ 249 and 250.

1. Multiply $\sqrt{-2}$ by $\sqrt{-3}$.

$$\sqrt{-2} \times \sqrt{-3} = \sqrt{2}\,\sqrt{-1} \times \sqrt{3}\,\sqrt{-1} \text{ (§ 249)}$$
$$= \sqrt{2}\,\sqrt{3}\,(\sqrt{-1})^2 = \sqrt{6}\,(-1) \text{ (§ 250)} = -\sqrt{6}, \textit{ Ans.}$$

2. Multiply together $\sqrt{-9}$, $\sqrt{-16}$, and $\sqrt{-25}$.

$$\begin{aligned}\sqrt{-9}\times\sqrt{-16}\times\sqrt{-25} &= 3\sqrt{-1}\times 4\sqrt{-1}\times 5\sqrt{-1}\\ &= 60(\sqrt{-1})^3 = 60(-\sqrt{-1})\ (\S\ 250)\\ &= -60\sqrt{-1},\ \textit{Ans.}\end{aligned}$$

3. Multiply $2\sqrt{-2}+\sqrt{-5}$ by $\sqrt{-2}-3\sqrt{-5}$.

$$\begin{array}{l}2\sqrt{-2}+\ \sqrt{-5}\\ \underline{\ \sqrt{-2}-3\sqrt{-5}\ }\\ 2(-2)+\ \sqrt{2}\sqrt{5}(\sqrt{-1})^2\\ \qquad\ \ \underline{-6\sqrt{2}\sqrt{5}(\sqrt{-1})^2-3(-5)}\\ \ -4-5\sqrt{10}(-1)\qquad\quad +15 = 11+5\sqrt{10},\ \textit{Ans.}\end{array}$$

Note. It should be remembered that to multiply an imaginary square root by itself simply removes the radical sign; thus,

$$\sqrt{-2}\times\sqrt{-2} = -2.$$

EXAMPLES.

Multiply the following:

4. $\sqrt{-49}$ by $\sqrt{-4}$.
5. $\sqrt{-9a^2}$ by $-\sqrt{-16a^2}$.
6. $\sqrt{-6}$ by $\sqrt{-10}$.
7. $-\sqrt{-a}$ by $-\sqrt{-b}$.
8. $-\sqrt{-27}$ by $\sqrt{-12}$.
9. $-\sqrt{-72}$ by $-\sqrt{-50}$.
10. $2-5\sqrt{-1}$ by $3+4\sqrt{-1}$.
11. $8+\sqrt{-2}$ by $7-5\sqrt{-2}$.
12. $4\sqrt{-5}-7\sqrt{-3}$ by $2\sqrt{-5}-\sqrt{-3}$.
13. $2\sqrt{-a}+3\sqrt{-b}$ by $4\sqrt{-a}-6\sqrt{-b}$.
14. $\sqrt{-x^2}$, $\sqrt{-y^2}$, and $\sqrt{-z^2}$.
15. $\sqrt{-8}$, $-\sqrt{-18}$, and $\sqrt{-32}$.
16. $\sqrt{-10}+5\sqrt{-5}$ by $3\sqrt{-3}+\sqrt{-6}$.
17. $\sqrt{-2}+\sqrt{-3}$ by $\sqrt{-8}-\sqrt{-12}$.

18. $\sqrt{-1}$, $\sqrt{-36}$, $\sqrt{-64}$, and $\sqrt{-100}$.

19. $\sqrt{-2}$, $\sqrt{-3}$, $-\sqrt{-5}$, and $\sqrt{-7}$.

Expand the following by the rules of §§ 78, 79, or 80.

20. $(1+\sqrt{-3})^2$.

21. $(5\sqrt{-2}+2\sqrt{-6})^2$.

22. $(4-\sqrt{-5})^2$.

23. $(2\sqrt{-3}-3\sqrt{-2})^2$.

24. $(x+\sqrt{-y})(x-\sqrt{-y})$.

25. $(5+6\sqrt{-1})(5-6\sqrt{-1})$.

26. $(3\sqrt{-a}+2\sqrt{-b})(3\sqrt{-a}-2\sqrt{-b})$.

27. $(7\sqrt{-2}+4\sqrt{-3})(7\sqrt{-2}-4\sqrt{-3})$.

28. $(\sqrt{-2}+2\sqrt{-1})^2+(\sqrt{-2}-2\sqrt{-1})^2$.

Reduce each of the following to an equivalent fraction having a rational denominator:

29. $\dfrac{2}{1+\sqrt{-3}}$.

30. $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}$.

31. $\dfrac{5\sqrt{-2}+4\sqrt{-6}}{5\sqrt{-2}-4\sqrt{-6}}$.

32. $\dfrac{\sqrt{-5}-6\sqrt{-3}}{3\sqrt{-5}+2\sqrt{-3}}$.

Expand the following by the rule of § 188:

33. $(1+\sqrt{-1})^3$.

34. $(2-\sqrt{-3})^3$.

253. Division of Imaginary Numbers.

1. Divide $\sqrt{-12}$ by $\sqrt{-3}$.

We have, $\dfrac{\sqrt{-12}}{\sqrt{-3}}=\dfrac{\sqrt{12}\sqrt{-1}}{\sqrt{3}\sqrt{-1}}=\dfrac{\sqrt{12}}{\sqrt{3}}=\sqrt{4}=2$, *Ans.*

2. Divide $\sqrt{10}$ by $\sqrt{-2}$.

$$\frac{\sqrt{10}}{\sqrt{-2}}=\frac{-\sqrt{10}(-1)}{\sqrt{2}\sqrt{-1}}=\frac{-\sqrt{10}(\sqrt{-1})^2}{\sqrt{2}\sqrt{-1}}(\S\,250)=-\sqrt{5}\sqrt{-1}=-\sqrt{-5},$$

Ans.

EXAMPLES.

Divide the following:

3. $\sqrt{-25}$ by $\sqrt{-5}$.
4. $\sqrt{-32}$ by $\sqrt{-8}$.
5. $\sqrt{42}$ by $\sqrt{-6}$.
6. $\sqrt{63}$ by $-\sqrt{-7}$.
7. $\sqrt{-ab}$ by $\sqrt{-bc}$.
8. $-\sqrt{a}$ by $\sqrt{-a^2}$.
9. $\sqrt[4]{-75}$ by $\sqrt[4]{-3}$.
10. $-\sqrt[4]{-18}$ by $\sqrt[4]{-2}$.
11. $\sqrt[6]{-108}$ by $\sqrt[6]{-4}$.
12. $-\sqrt[6]{-40}$ by $-\sqrt[6]{-5}$.

SOLUTION OF EQUATIONS CONTAINING RADICALS.

254. 1. Solve the equation $\sqrt{x^2-5}-x=-1$.

Transposing $-x$, $\sqrt{x^2-5}=x-1$.

Squaring both members, $x^2-5=x^2-2x+1$.

Transposing and uniting terms, $2x=6$.

Whence, $x=3$, *Ans.*

2. Solve the equation $\sqrt{2x+14}+\sqrt{2x+35}=7$.

Transposing $\sqrt{2x+14}$, $\sqrt{2x+35}=7-\sqrt{2x+14}$.

Squaring both members, $2x+35=49-14\sqrt{2x+14}+2x+14$.

Transposing and uniting terms,

$$14\sqrt{2x+14}=28.$$

Or, $\sqrt{2x+14}=2$.

Squaring both members, $2x+14=4$.

$$2x=-10.$$

Whence, $x=-5$, *Ans.*

From the above examples, we derive the following rule:

Transpose the terms of the equation so that a radical term may stand alone in one member; then raise both members to a power of the same degree as the radical.

If radical terms still remain, repeat the operation.

Note. The equation should be simplified as much as possible before performing the involution.

EXAMPLES.

Solve the following:

3. $\sqrt{3x-5}-2=0.$ 7. $\sqrt{x+4}+\sqrt{x}=3.$

4. $\sqrt[3]{6x+9}+8=5.$ 8. $\sqrt[3]{8x^3-12x^2}+1=2x.$

5. $\sqrt{9x^2+5}-3x=1.$ 9. $\sqrt{5x+10}-\sqrt{5x}=2.$

6. $\sqrt{x}-\sqrt{x-12}=2.$ 10. $\sqrt{x+11}+\sqrt{x+6}=5.$

11. $\dfrac{3}{\sqrt{3-x}}-\dfrac{x}{\sqrt{6-x}}=\sqrt{6-x}.$

12. $\sqrt{2x-1}+\sqrt{2x+4}=5.$ 15. $\sqrt{x+5}+\sqrt{x-3}=2\sqrt{x}.$

13. $\sqrt{x}+\sqrt{x+4}=\dfrac{2}{\sqrt{x}}.$ 16. $\dfrac{2\sqrt{x}-3}{3\sqrt{x}+2}=\dfrac{4\sqrt{x}-4}{6\sqrt{x}+1}.$

14. $\sqrt{x-6}+\sqrt{x}=\dfrac{3}{\sqrt{x-6}}.$ 17. $\dfrac{\sqrt{3x+1}+\sqrt{3x}}{\sqrt{3x+1}-\sqrt{3x}}=4.$

18. $\sqrt{x^2-5x-2}+\sqrt{x^2+3x+6}=4.$

19. $\sqrt{x}-\sqrt{x-5}=\dfrac{10}{\sqrt{x-5}}.$ 21. $\dfrac{\sqrt{x+a}-\sqrt{x-a}}{\sqrt{x+a}+\sqrt{x-a}}=2.$

20. $\sqrt{x+15}-\sqrt{x+3}=2\sqrt{x}.$ 22. $\sqrt{x+a}+\sqrt{x-a}=2b$

23. $\sqrt{x+2a}-\sqrt{x-3a}=\dfrac{a}{\sqrt{x-3a}}.$

24. $\sqrt{x+6}+\sqrt{x+9}=\sqrt{4x+29}.$

25. $\sqrt{x+a}+\sqrt{x+4a}=2\sqrt{x+2a}.$

26. $\sqrt{cx+ab}+\sqrt{cx-ab}=\sqrt{4cx-2ab}.$

27. $\sqrt{x}+\sqrt{a-\sqrt{ax+x^2}}=\sqrt{a}.$

28. $\sqrt{(x+2a\sqrt{4x+3a^2})}=\sqrt{x}-2a.$

29. $2(x+a)(x+\sqrt{x^2-a^2})=a^2.$

30. $\sqrt{x+1}+\sqrt{x+5}=\sqrt{x+2}+\sqrt{x+3}.$

31. $\sqrt{a-x}+\sqrt{b-x}=\sqrt{a+3b-4x}.$

XXII. QUADRATIC EQUATIONS.

255. A **Quadratic Equation** is an equation of the second degree (§ 158).

A **Pure Quadratic Equation** is a quadratic equation involving only the square of the unknown quantity; as $2x^2 = 5$.

An **Affected Quadratic Equation** is a quadratic equation involving both the square and the first power of the unknown quantity; as $2x^2 - 3x - 5 = 0$.

PURE QUADRATIC EQUATIONS.

256. A pure quadratic equation may be solved by reducing it, if necessary, to the form $x^2 = a$, and then extracting the square root of both members.

1. Solve the equation $3x^2 + 7 = \frac{5x^2}{4} + 35$.

Clearing of fractions, $12x^2 + 28 = 5x^2 + 140$.

Transposing and uniting terms, $7x^2 = 112$.

Or, $x^2 = 16$.

Extracting the square root of both members, we have

$x = \pm 4$, *Ans.*

Note 1. The sign ± is placed before the result, because the square root of a number is either positive or negative (§ 192).

2. Solve the equation $7x^2 - 5 = 5x^2 - 13$.

Transposing and uniting terms, $2x^2 = -8$.

Or, $x^2 = -4$.

Whence, $x = \pm\sqrt{-4} = \pm 2\sqrt{-1}$, *Ans.*

Note 2. In Ex. 2 the values of x are *imaginary* (§ 248); it is impossible to find any real values of x which will satisfy the given equation.

EXAMPLES.

Solve the following equations:

3. $3x^2 - 26 = 9x^2 - 80.$ 5. $3(x+1) - x(x-1) = 4x.$

4. $\frac{7}{3x^2} - \frac{11}{9x^2} = \frac{5}{8}.$ 6. $\frac{x}{3} + \frac{3}{x} = \frac{x}{12} + \frac{12}{x}.$

7. $(2x-3)(x+7) + (2x+3)(x-7) = 58.$

8. $5(x+a)(x-a) + 4ax = (x+2a)^2.$

9. $(3x+2)(4x-5) - (5x+3)(6x-5) + 45 = 0.$

10. $\frac{2x^2+4}{5} - \frac{3x^2-7}{3} = \frac{11}{15}.$ 12. $x + \sqrt{a^2+x^2} = \frac{5a^2}{\sqrt{a^2+x^2}}.$

11. $\sqrt{10+x} - \sqrt{10-x} = 2.$ 13. $\sqrt{2x+8} + 2\sqrt{x+5} = 2.$

14. $\frac{x^2+x+1}{x-1} - \frac{x^2-x+1}{x+1} = 6.$

15. $\frac{3x^2-2}{5} - \frac{5x^2+3}{10} - \frac{4x^2-4}{25} = 0.$

16. $\frac{9x^2-2}{6} = \frac{3x^2}{2} - \frac{6x^2+1}{9x^2+7}.$

17. $\frac{x+4}{x-4} + \frac{x-4}{x+4} = \frac{10}{3}.$ 18. $\frac{x^4-3x^2+4}{3x^4+2x^2-4} = \frac{x^2-3}{3x^2+2}.$

19. $x\sqrt{x^2+12} + x\sqrt{x^2+6} = 3.$

20. $\frac{x+a}{x-b} + \frac{x-a}{x+b} = 1 + \frac{a^2+2b^2}{x^2-b^2}.$

21. $\sqrt{1+x+x^2} - \sqrt{1-x+x^2} = \sqrt{6}.$

22. $\frac{x+a}{x-a} + \frac{x-a}{x+a} = \frac{a+b}{a-b} + \frac{a-b}{a+b}.$

(First add the fractions in the first member; then the fractions in the second member.)

AFFECTED QUADRATIC EQUATIONS.

257. An affected quadratic equation may be solved by reducing it, if necessary, to the form $x^2+px=q$.

We then add to both members such an expression as will make the first member a perfect trinomial square (§ 96); an operation which is termed *completing the square.*

258. First Method of Completing the Square.

Example. Solve the equation $x^2+3x=4$.

A trinomial is a perfect square when its first and third terms are perfect squares and positive, and the second term plus (or minus) twice the product of their square roots (§ 96).

Then, the square root of the third term is equal to the second term divided by twice the square root of the first.

Hence, the *square root* of the expression which must be added to x^2+3x to make it a perfect square is $\frac{3x}{2x}$, or $\frac{3}{2}$.

Adding to both members the square of $\frac{3}{2}$, we have

$$x^2+3x+\left(\frac{3}{2}\right)^2=4+\frac{9}{4}=\frac{25}{4}.$$

Extracting the square root of both members (§ 97),

$$x+\frac{3}{2}=\pm\frac{5}{2}. \quad \text{(See Note 1, § 256.)}$$

Transposing $\frac{3}{2}$, $\quad x=-\frac{3}{2}+\frac{5}{2}$, or $-\frac{3}{2}-\frac{5}{2}$.

Whence, $\quad x=1$ or -4, *Ans.*

From the above example, we derive the following rule:

Reduce the equation to the form $x^2+px=q$.

Complete the square by adding to both members the square of half the coefficient of x.

Extract the square root of both members, and solve the simple equations thus formed.

259. 1. Solve the equation $3x^2 - 8x = -4$.

Dividing by 3, $x^2 - \frac{8x}{3} = -\frac{4}{3}$;
which is in the form $x^2 + px = q$.

Adding to both members the square of $\frac{4}{3}$, we have

$$x^2 - \frac{8x}{3} + \left(\frac{4}{3}\right)^2 = -\frac{4}{3} + \frac{16}{9} = \frac{4}{9}.$$

Extracting the square root, $x - \frac{4}{3} = \pm\frac{2}{3}$.

Whence, $x = \frac{4}{3} \pm \frac{2}{3} = 2$ or $\frac{2}{3}$, *Ans.*

If the coefficient of x^2 is negative, the sign of each term must be changed.

2. Solve the equation $-9x^2 - 21x = 10$.

Dividing by -9, $x^2 + \frac{7x}{3} = -\frac{10}{9}$.

Adding to both members the square of $\frac{7}{6}$, we have

$$x^2 + \frac{7x}{3} + \left(\frac{7}{6}\right)^2 = -\frac{10}{9} + \frac{49}{36} = \frac{9}{36}.$$

Extracting the square root, $x + \frac{7}{6} = \pm\frac{3}{6}$.

Whence, $x = -\frac{7}{6} \pm \frac{3}{6} = -\frac{2}{3}$ or $-\frac{5}{3}$, *Ans.*

EXAMPLES.

Solve the following equations:

3. $x^2 + 6x = 7$.
4. $x^2 - 4x = 32$.
5. $x^2 + 11x = -18$.
6. $x^2 - 13x = -30$.
7. $x^2 + x = 30$.
8. $3x^2 - 7x = -2$.
9. $4x^2 - 3x = 7$.
10. $2x^2 + 11x = -5$.
11. $2x^2 + 9x - 5 = 0$.
12. $5x^2 + 8 = 22x$.
13. $20 - 27x = -9x^2$.
14. $7 - 10x - 8x^2 = 0$.
15. $12 + 16x - 3x^2 = 0$.
16. $6x^2 + 4 = -11x$.

260. If the coefficient of x^2 is a perfect square, it is convenient to complete the square directly by the principle stated in § 258; that is, *by adding to both members the square of the quotient obtained by dividing the coefficient of x by twice the square root of the coefficient of* x^2.

1. Solve the equation $9x^2 - 5x = 4$.

Dividing 5 by twice the square root of 9, the quotient is $\frac{5}{6}$.
Adding to both members the square of $\frac{5}{6}$, we have

$$9x^2 - 5x + \left(\frac{5}{6}\right)^2 = 4 + \frac{25}{36} = \frac{169}{36}.$$

Extracting the square root, $3x - \frac{5}{6} = \pm\frac{13}{6}$.

Transposing, $3x = \frac{5}{6} \pm \frac{13}{6} = 3 \text{ or } -\frac{4}{3}$.

Whence, $x = 1 \text{ or } -\frac{4}{9}$, *Ans.*

If the coefficient of x^2 is not a perfect square, it may be made so by multiplication.

2. Solve the equation $8x^2 - 15x = 2$.

Multiplying each term by 2, $16x^2 - 30x = 4$.
Dividing 30 by twice the square root of 16, the quotient is $\frac{30}{8}$, or $\frac{15}{4}$.
Adding to both members the square of $\frac{15}{4}$, we have

$$16x^2 - 30x + \left(\frac{15}{4}\right)^2 = 4 + \frac{225}{16} = \frac{289}{16}.$$

Extracting the square root, $4x - \frac{15}{4} = \pm\frac{17}{4}$.

Transposing, $4x = \frac{15}{4} \pm \frac{17}{4} = 8 \text{ or } -\frac{1}{2}$.

Whence, $x = 2 \text{ or } -\frac{1}{8}$, *Ans.*

Note. If the coefficient of x^2 is negative, the sign of each term must be changed.

EXAMPLES.

Solve the following equations:

3. $4x^2 + 7x = 2.$
4. $16x^2 + 32x = -15.$
5. $9x^2 - 11x = -2.$
6. $8x^2 + 2x = 3.$
7. $5x^2 + 16x = -3.$
8. $36x^2 - 36x = -5.$
9. $64x^2 + 48x = 7.$
10. $49x^2 - 7x = 12.$
11. $25x^2 + 25x + 6 = 0.$
12. $12x^2 + 8x = -1.$
13. $32x^2 + 1 = -12x.$
14. $28 + 5x - 3x^2 = 0.$
15. $x + 1 = 20x^2.$
16. $4 + 3x - 27x^2 = 0.$

261. Second Method of Completing the Square.

Every affected quadratic can be reduced to the form

$$ax^2 + bx = c.$$

Multiplying both members by $4a$, we have

$$4a^2x^2 + 4abx = 4ac.$$

Completing the square by adding to both members the square of $\frac{4ab}{2 \times 2a}$ (§ 260), or b, we obtain

$$4a^2x^2 + 4abx + b^2 = b^2 + 4ac.$$

Extracting the square root,

$$2ax + b = \pm\sqrt{b^2 + 4ac}.$$

Transposing, $$2ax = -b \pm \sqrt{b^2 + 4ac}.$$

Whence, $$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}.$$

From the above example, we derive the following rule:

Reduce the equation to the form $ax^2 + bx = c$.

Multiply both members by four times the coefficient of x^2, *and add to each the square of the coefficient of* x *in the given equation.*

Extract the square root of both members, and solve the simple equation thus formed.

The advantage of this method over the preceding is in avoiding fractions in completing the square.

262. 1. Solve the equation $2x^2 - 7x = -3$.

Multiplying both members by 4×2, or 8,

$$16x^2 - 56x = -24.$$

Adding to both members the square of 7, we have

$$16x^2 - 56x + 7^2 = -24 + 49 = 25.$$

Extracting the square root, $4x - 7 = \pm 5$.

$$4x = 7 \pm 5 = 12 \text{ or } 2.$$

Whence, $x = 3$ or $\frac{1}{2}$, *Ans.*

If the coefficient of x in the given equation is *even*, fractions may be avoided, and the rule modified, as follows:

Multiply both members by the coefficient of x^2, and add to each the square of half the coefficient of x in the given equation.

2. Solve the equation $15x^2 + 28x = 32$.

Multiplying both members by 15,

$$15^2x^2 + 15(28x) = 480.$$

Adding to both members the square of 14, we have

$$15^2x^2 + 15(28x) + 14^2 = 480 + 196 = 676.$$

Extracting the square root, $15x + 14 = \pm 26$.

$$15x = -14 \pm 26 = 12 \text{ or } -40.$$

Whence, $x = \frac{4}{5}$ or $-\frac{8}{3}$, *Ans.*

EXAMPLES.

Solve the following equations:

3. $x^2 - 7x = 30$.
4. $2x^2 + 5x = 18$.
5. $3x^2 - 2x = 33$.
6. $8x^2 + 14x = -3$.
7. $10x^2 + 7x = -1$.
8. $5x^2 - 2x = 72$.

9. $4x^2 - 7x = -3.$
10. $6x^2 - 11x = 10$
11. $4x^2 + 24x + 35 = 0.$
12. $4x + 4 = 15x^2.$
13. $4 - 15x - 4x^2 = 0.$
14. $6x^2 + 17x = -10.$
15. $5x^2 + 15 = 28x.$
16. $9x^2 = 32x - 15.$
17. $3 - 5x - 12x^2 = 0.$
18. $9x^2 + 15x + 4 = 0.$

MISCELLANEOUS EXAMPLES.

263. The following equations may be solved by either of the preceding methods, preference being given to the one best adapted to the example under consideration.

1. $\frac{x^2}{3} - \frac{x}{2} = \frac{35}{6}.$
2. $\frac{1}{2} - \frac{5}{6x^2} = -\frac{7}{12x}.$
3. $\frac{1}{8x^2} - \frac{13}{24x} = -\frac{1}{2}.$
4. $\frac{x}{4} + \frac{4}{x} = -\frac{29}{10}.$
5. $(3x + 2)(2x + 3) = (x - 3)(2x - 4).$
6. $9(x - 1)^2 - 4(x - 2)^2 = 44.$
7. $4(x - 1)(2x - 1) + 4(2x - 1)(3x - 1) + 4(3x - 1)(4x - 1) = 53x^2.$
8. $\frac{30}{x} - \frac{30}{x + 1} = 1.$
9. $\frac{3}{x - 6} - \frac{2}{x - 5} = 1.$
10. $\frac{x + 2}{x - 1} - \frac{4 - x}{2x} = \frac{7}{3}.$
11. $\frac{x + 1}{2x + 1} + \frac{2x + 1}{3x + 1} = \frac{17}{12}.$
12. $(2x + 1)^2 - (3x - 2)^2 - (x + 1)^2 = 0.$
13. $\sqrt{6 + 10x - 3x^2} = 2x - 3.$
14. $\frac{2 - 3x}{4} - \frac{4 - x}{x - 2} = \frac{11}{4}.$
15. $(x - 3)^3 - (x + 2)^3 = -65.$
16. $\sqrt{x - 1} + \sqrt{3x + 3} = 4.$
17. $\frac{x}{3} - \frac{x^2 - 6}{3(x + 4)} = \frac{6}{x}.$
18. $\sqrt{x + 2} + \sqrt{3x + 4} = 8.$
19. $\frac{\sqrt{12 - x}}{5} = \frac{3}{2 + \sqrt{12 - x}}.$

20. $\sqrt[3]{x^3+8x^2+16x-1}=x+3.$

21. $1+\frac{7x}{3x+1}+\frac{2x^2}{(3x+1)(7x+1)}=0.$

22. $\frac{1}{x-6}-\frac{4}{3(x-1)}=-\frac{2}{3x}.$

23. $\frac{4x-9}{4x-3}-\frac{2x-3}{2x}=9.$

24. $\frac{x+2}{x-2}+\frac{x-2}{x+2}=\frac{6x+16}{3x}.$

25. $\sqrt{5+x}+\sqrt{5-x}=\frac{12}{\sqrt{5-x}}.$

26. $\sqrt{x+3}-\sqrt{x+8}=-5\sqrt{x}.$

27. $\frac{1}{1-x^2}+\frac{1}{1+x}-\frac{1}{1-x}=-\frac{7}{8}.$

28. $3-\frac{1}{x+2}-\frac{3}{2(2x-3)}=\frac{5}{(x+2)(2x-3)}.$

29. $\frac{3x-1}{7-x}-\frac{5-4x}{2x+1}=3.$

30. $\frac{3x-6}{5-x}=\frac{7}{2}-\frac{11-2x}{2(5-2x)}.$

31. $\frac{3-2x}{2+x}-\frac{2+3x}{2-x}=\frac{1}{3}+\frac{16x+x^2}{x^2-4}.$

32. $\frac{x+1}{x-1}+\frac{x+2}{x-2}=\frac{2x+13}{x+1}.$

33. $\sqrt{2x^2+9x+9}+\sqrt{2x^2+7x+5}=\sqrt{2}.$

34. $\sqrt{3x+1}-\sqrt{4x+5}+\sqrt{x-4}=0.$

35. $\frac{1}{(3x+1)(1-5x)}-\frac{15x}{2(1-5x)(7x+1)}=\frac{x}{(3x+1)(7x+1)}.$

36. $\frac{\sqrt{x}}{\sqrt{x+2}}-\frac{\sqrt{x+2}}{\sqrt{x}}=\frac{5}{6}.$

37. $\sqrt{5-2x}+\sqrt{15-3x}=\sqrt{26-5x}.$

38. $\dfrac{x-1}{x+3}+\dfrac{x+3}{x-1}-\dfrac{2(x+4)}{x-2}=0.$

264. Solution of Literal Quadratic Equations.

For the solution of literal affected quadratic equations, the methods of § 262 will be found in general the most convenient.

1. Solve the equation $x^2+ax-bx-ab=0$.

The equation may be written

$$x^2+(a-b)x=ab.$$

Multiplying both members by 4 times the coefficient of x^2,

$$4x^2+4(a-b)x=4ab.$$

Adding to both members the square of $a-b$,

$$\begin{aligned}4x^2+4(a-b)x+(a-b)^2&=4ab+a^2-2ab+b^2\\&=a^2+2ab+b^2.\end{aligned}$$

Extracting the square root,

$$2x+(a-b)=\pm(a+b).$$

$$2x=-(a-b)\pm(a+b).$$

Therefore, $2x=-a+b+a+b=2b,$

or $2x=-a+b-a-b=-2a.$

Whence, $x=b$ or $-a$, *Ans.*

Note. If several terms contain the same power of x, the coefficient of that power should be enclosed in a parenthesis, as shown in Ex. 1.

2. Solve the equation $(m-1)x^2-2m^2x=-4m^2$.

Multiplying both members by $m-1$,

$$(m-1)^2x^2-2m^2(m-1)x=-4m^2(m-1).$$

Adding to both members the square of m^2,

$$(m-1)^2x^2-2m^2(m-1)x+m^4=m^4-4m^3+4m^2.$$

Extracting the square root,

$$(m-1)x-m^2=\pm(m^2-2m).$$

$$(m-1)x=m^2+m^2-2m \text{ or } m^2-m^2+2m$$

$$=2m(m-1) \text{ or } 2m.$$

Whence, $$x=2m \text{ or } \frac{2m}{m-1},\ \textit{Ans.}$$

EXAMPLES.

Solve the following equations:

3. $x^2-4ax=9b^2-4a^2.$

4. $x^2+2mx=2m+1.$

5. $x^2-(a-1)x=a.$

6. $x^2+ax+bx+ab=0.$

7. $x^2-m^2x-m^3x=-m^5.$

8. $acx^2+bcx-adx=bd.$

9. $x^2-2ax-12x=3a^2-16a-35.$

10. $(a-b)x^2-(a+b)x=-2b.$

11. $(a-x)(a^2+b^2+ax)=a^3+bx^2.$

12. $\frac{1+a}{1-ax}+\frac{1-a}{1+ax}=1.$

13. $\frac{b}{x-a}+\frac{a}{x-b}=2.$

14. $(x+2a)^3-(x-3a)^3=65a^3.$

15. $(1-a^2)(x+a)-2a(1-x^2)=0.$

16. $\sqrt{(a-2b)x+8ab}=x+4b.$

17. $6x^2-(5a+b)x=-a^2-ab+2b^2.$

18. $\sqrt{x+7a}+\sqrt{x-2a}=\sqrt{5x+2a}.$

19. $\frac{x-a}{x+a}+\frac{x+a}{x-a}-\frac{5ax-3a-2}{a^2-x^2}=0.$

20. $\sqrt{x-12ab}=\frac{9a^2-b^2}{\sqrt{x}}.$

21. $\frac{x^2+1}{x}=\frac{2(a^2+b^2)}{a^2-b^2}.$

22. $\frac{x+a}{x+b}+\frac{x+b}{x+a}=\frac{5}{2}.$

23. $\sqrt{2x-4a}+\sqrt{5x+3a}=\frac{3x-a}{\sqrt{2x-4a}}.$

24. $x^2-(a-b)x=(a-c)(b-c)$.

25. $\sqrt{3x+2a}-\sqrt{4x-6a}=\sqrt{2a}$.

26. $\left(\frac{x-m}{x+m}\right)^2-7\left(\frac{x-m}{x+m}\right)+12=0$.

27. $\frac{x+\sqrt{12a-x}}{x-\sqrt{12a-x}}=\frac{\sqrt{a}+1}{\sqrt{a}-1}$.

28. $(a-b)x^2+(b-c)x+(c-a)=0$.

29. $(a^4-1)x^2-2(a^4+1)x=-a^4+1$.

30. $\frac{x^2+1}{x}=\frac{a-b}{c}+\frac{c}{a-b}$.

31. $\frac{1}{x-a}+\frac{1}{x-b}=\frac{1}{a}+\frac{1}{b}$.

32. $(c+a-2b)x^2+(a+b-2c)x+b+c-2a=0$.

265. Solution of Quadratic Equations by a Formula.

It was shown in § 261 that, if $ax^2+bx=c$, then

$$x=\frac{-b\pm\sqrt{b^2+4ac}}{2a}. \qquad (1)$$

This result may be used as a *formula* for the solution of any quadratic equation in the form $ax^2+bx=c$.

1. Solve the equation $2x^2+5x=18$.

In this case, $a=2$, $b=5$, and $c=18$; substituting in (1),

$$x=\frac{-5\pm\sqrt{25+144}}{4}=\frac{-5\pm\sqrt{169}}{4}=\frac{-5\pm13}{4}=2 \text{ or } -\frac{9}{2}, \textit{ Ans.}$$

2. Solve the equation $110x^2-21x=-1$.

In this case, $a=110$, $b=-21$, and $c=-1$; substituting in (1),

$$x=\frac{21\pm\sqrt{441-440}}{220}=\frac{21\pm1}{220}=\frac{1}{10} \text{ or } \frac{1}{11}, \textit{ Ans.}$$

Note. Particular attention must be paid to the *signs* of the coefficients in making the substitution.

EXAMPLES.

Solve the following equations:

3. $2x^2 + x = 6.$
4. $x^2 - 5x = 36.$
5. $x^2 + 14x + 48 = 0.$
6. $5x^2 - 13x = -6.$
7. $6x^2 - x = 5.$
8. $3x^2 - 7x = 20.$
9. $4x^2 - 21x = -27.$
10. $28x^2 + 16x = -1.$
11. $8x^2 + 41x + 5 = 0.$
12. $16x^2 + 16x - 5 = 0.$
13. $30x - 8 = 25x^2.$
14. $12x^2 + 7 = -25x.$
15. $2 - 3x - 54x^2 = 0.$
16. $3 + 14x - 24x^2 = 0.$

266. Solution of Equations by Factoring.

Let it be required to solve the equation

$$(x - 3)(2x + 5) = 0.$$

It is evident that the equation will be satisfied when x has such a value that *one* of the factors of the first member is equal to zero; for if any factor of a product is equal to zero, the product is equal to zero.

Hence, the equation will be satisfied when x has such a value that either

$$x - 3 = 0, \qquad (1)$$

or

$$2x + 5 = 0. \qquad (2)$$

Solving (1) and (2), we have $x = 3$ or $-\frac{5}{2}$.

It will be observed that the roots are obtained by *placing the factors of the first member separately equal to zero, and solving the resulting equations.*

267. 1. Solve the equation $x^2 - 5x - 24 = 0.$

Factoring the first member, $(x - 8)(x + 3) = 0.$ (§ 100)

Placing the factors separately equal to zero (§ 266), we have

$$x - 8 = 0, \text{ and } x + 3 = 0.$$

Whence, $x = 8$ or -3, *Ans.*

2. Solve the equation $2x^2 - x = 0$.

Factoring the first member, $x(2x-1)=0$.

Placing the factors separately equal to zero,

$$x = 0, \text{ and } 2x - 1 = 0.$$

Whence, $x = 0$ or $\frac{1}{2}$, *Ans.*

3. Solve the equation $x^3 + 4x^2 - x - 4 = 0$.

Factoring the first member, $(x+4)(x^2-1)=0$. (§ 93)

Therefore, $x + 4 = 0$, and $x^2 - 1 = 0$.

Whence, $x = -4$ or ± 1, *Ans.*

4. Solve the equation $x^3 - 1 = 0$.

Factoring the first member, $(x-1)(x^2+x+1)=0$. (§ 103)

Therefore, $x - 1 = 0$, and $x^2 + x + 1 = 0$.

Solving the equation $x - 1 = 0$, we have $x = 1$.

Solving the equation $x^2 + x + 1 = 0$, we have

$$x = \frac{-1 \pm \sqrt{1-4}}{2} \text{ (§ 265)} = \frac{-1 \pm \sqrt{-3}}{2}.$$

EXAMPLES.

Solve the following equations:

5. $x^2 + 3x - 28 = 0$.

6. $x^2 - 14x + 45 = 0$.

7. $x^2 + 11x + 24 = 0$.

8. $x^2 - 6x - 72 = 0$.

9. $5x^2 - 7x = 0$.

10. $3x^3 + 24x^2 = 0$.

11. $16x^3 - 9x = 0$.

12. $(2x+5)(9x^2-49) = 0$.

13. $12x^3 - 7x^2 - 10x = 0$.

14. $(x^2-8)(x^2+4) = 0$.

15. $(x-3)(2x^2+13x+20) = 0$.

16. $(x-3)(x+4)(x-5) - 60 = 0$.

17. $(x^2-9a^2)(2x^2+ax-a^2) = 0$.

18. $x^3+1=0.$

19. $x^3-27=0.$

20. $16x^4-81=0.$

21. $27x^3-64a^3=0.$

22. $8x^3+125=0.$

23. $x^6-64=0.$

24. $x^3-x^2+x-1=0.$

25. $\sqrt{x^2-\sqrt{2x+1}}=x-1$

26. $5x^3-x^2-125x+25=0.$

27. $8x^3+20x^2-18x-45=0.$

28. $4x^3+5x^2+72x+90=0.$

29. $\sqrt{a+x}+\sqrt{a-x}=\sqrt{x}.$

30. $\sqrt{a+\sqrt{x}}-\sqrt{a-\sqrt{x}}=\sqrt{x}.$

Note. The above examples are illustrations of the important principle that the degree of an equation indicates the number of its roots; thus, an equation of the third degree has three roots; of the fourth degree, four roots; etc.

It should be observed that the roots are not necessarily *unequal*; thus, the equation $x^2-2x+1=0$ may be written $(x-1)(x-1)=0$, and therefore its two roots are 1 and 1.

PROBLEMS.

268. 1. A man sold a watch for $21, and lost as many per cent as the watch cost dollars. What was the cost?

Let $x=$ the number of dollars the watch cost.

Then, $x=$ the per cent of loss,

and $x\times\frac{x}{100}$, or $\frac{x^2}{100}=$ the number of dollars lost.

By the conditions, $\frac{x^2}{100}=x-21.$

Solving, $x=30$ or $70.$

Then, the cost of the watch was either $30 or $70; for either of these answers satisfies the conditions of the problem.

2. A farmer bought some sheep for $72. If he had bought 6 more for the same money, they would have cost him $1 apiece less. How many did he buy?

Let $x =$ the number bought.

Then, $\frac{72}{x} =$ the number of dollars paid for one,

and $\frac{72}{x+6} =$ the number of dollars paid for one if there had been 6 more.

By the conditions, $\frac{72}{x} = \frac{72}{x+6} + 1.$

Solving, $x = 18$ or $-24.$

Only the *positive* value of x is admissible, for the negative value does not satisfy the conditions of the problem.

Therefore, the number of sheep was 18.

Note 1. In solving problems which involve quadratics, there will usually be two values of the unknown quantity; and those values only should be retained as answers which satisfy the conditions of the problem.

Note 2. If, in the enunciation of the problem, the words "6 more" had been changed to "6 *fewer*," and "$1 apiece less" to "$1 apiece *more*," we should have found the answer 24.

In many cases where the solution of a problem gives a negative result, the wording may be changed so as to form an analogous problem to which the absolute value of the negative result is an answer.

3. I bought a lot of flour for $126; and the number of dollars per barrel was $\frac{2}{7}$ the number of barrels. How many barrels were purchased, and at what price?

4. Divide the number 18 into two parts, the sum of whose squares shall be 170.

5. Find two numbers whose difference is 7, and whose sum multiplied by the greater is 400.

6. Find three consecutive numbers whose sum is equal to the product of the first two.

7. Divide the number 20 into two parts such that one is the square of the other.

8. Find two numbers whose sum is 7, and the sum of whose cubes is 133.

9. Find four consecutive numbers such that if the first two be taken as the digits of a number, that number is equal to the product of the other two.

10. A merchant bought a quantity of flour for $108. If he had bought 9 barrels more for the same money, he would have paid $2 less per barrel. How many barrels did he buy, and at what price?

11. A farmer bought a number of sheep for $378. Having lost 6, he sold the remainder for $10 a head more than they cost him, and gained $42. How many did he buy?

12. A merchant sold a quantity of wheat for $56, and gained as many per cent as the wheat cost dollars. What was the cost of the wheat?

13. If the product of three consecutive numbers be divided by each of them in turn, the sum of the three quotients is 74. What are the numbers?

14. A crew can row 8 miles down stream and back again in $4\frac{4}{5}$ hours; if the rate of the stream is 4 miles an hour, find the rate of the crew in still water.

15. A certain farm is a rectangle, whose length is three times its width. If its length should be increased by 20 rods, and its width by 8 rods, its area would be trebled. Of how many square rods does the farm consist?

16. A man travels 9 miles by train. He returns by a train which runs 9 miles an hour faster than the first, and accomplishes the entire journey in 35 minutes. Required the rates of the trains.

17. The area of a rectangular field is 216 square rods, and its perimeter is 60 rods. Find its length and width.

18. At what price per dozen are eggs selling when, if the price were raised 5 cents per dozen, one would receive twelve less for a dollar?

19. A merchant sold goods for $18.75, and lost as many per cent as the goods cost dollars. What was the cost?

20. A man travelled by coach 6 miles, and returned on foot at a rate 5 miles an hour less than that of the coach. He was 50 minutes longer in returning than in going. What was the rate of the coach?

21. A square picture is surrounded by a frame. The side of the picture exceeds by an inch the width of the frame; and the number of square inches in the frame exceeds by 124 the number of inches in the perimeter of the picture. Find the area of the picture, and the width of the frame.

22. The circumference of the fore-wheel of a carriage is less by 4 feet than that of the hind-wheel. In travelling 1200 feet, the fore-wheel makes 25 revolutions more than the hind-wheel. Find the circumference of each wheel.

23. A tank can be filled by two pipes running together in $3\frac{3}{4}$ hours. The larger pipe by itself will fill it sooner than the smaller by 4 hours. What time will each pipe separately take to fill it?

24. The telegraph poles along a certain railway are at equal intervals. If there were two more in each mile, the interval between the poles would be decreased by 20 feet. Find the number of poles in a mile.

25. A and B gained in trade $2100. A's money was in the firm 15 months, and he received in principal and gain $3900. B's money, which was $5000, was in the firm 12 months. How much money did A put into the firm?

26. If $2000 amounts to $2205, when put at compound interest for two years, the interest being compounded annually, what is the rate per cent per annum?

27. A man travelled 105 miles. If he had gone 4 miles more an hour, he would have performed the journey in $9\frac{1}{3}$ hours less time. How many miles an hour did he go?

6|486
9|81
9

x^2

86
76

27|486|18
27
216

28. The sum of $120 was divided between a certain number of persons. If each person had received $7 less, he would have received as many dollars as there were persons. Required the number of persons.

29. My income is $5000. After deducting a percentage for income tax, and then a percentage, less by one than that of the income tax, from the remainder, the income is reduced to $4656. Find the rate per cent of the income tax.

30. A man has two square lots of unequal size, together containing 13,325 square feet. If the lots were contiguous, it would require 510 feet of fence to embrace them in a single enclosure of six sides. Find the area of each lot.

31. A merchant has a cask full of wine, containing 36 gallons. He draws a certain number of gallons, and then fills the cask up with water. He then draws out the same number of gallons as before, and finds that there are 25 gallons of pure wine remaining in the cask. How many gallons did he draw each time?

32. A set out from C towards D at the rate of 5 miles an hour. After he had gone 32 miles, B set out from D towards C, and went every hour $\frac{1}{17}$ of the entire distance; and after he had travelled as many hours as he went miles in an hour, he met A. Required the distance from C to D.

33. A courier travels from P to Q in 12 hours. Another courier starts at the same time from a place 24 miles the other side of P, and arrives at Q at the same time as the first courier. The second courier finds that he takes half an hour less than the first to accomplish 12 miles. Find the distance from P to Q.

34. A man bought a number of $50 shares, when they were at a certain rate per cent premium, for $4800; and afterwards, when they were at the same rate per cent discount, sold them all but 30 for $2000. How many shares did he buy, and how much did he give apiece?

XXIII. EQUATIONS SOLVED LIKE QUADRATICS.

EQUATIONS IN THE QUADRATIC FORM.

269. An equation is said to be in the *quadratic form* when it is expressed in three terms, two of which contain the unknown quantity, and *the exponent of the unknown quantity in one of these terms is twice its exponent in the other;* as,

$$x^6 - 6x^3 = 16;$$

$$x^3 + x^{\frac{3}{2}} = 72; \text{ etc.}$$

270. Equations in the quadratic form may be readily solved by the rules for quadratics.

1. Solve the equation $x^6 - 6x^3 = 16$.

Completing the square by the rule of § 258,

$$x^6 - 6x^3 + 3^2 = 16 + 9 = 25.$$

Extracting the square root, $x^3 - 3 = \pm 5$.

Whence, $x^3 = 3 \pm 5 = 8 \text{ or } -2$.

Extracting the cube root, $x = 2 \text{ or } -\sqrt[3]{2}$, *Ans.*

Note 1. There are also four imaginary roots, which may be obtained by the method of § 267.

2. Solve the equation $2x + 3\sqrt{x} = 27$.

Since $\sqrt{x}$ is the same as $x^{\frac{1}{2}}$, this is in the quadratic form.
Multiplying by 8, and adding 3^2 to both members (§ 261),

$$16x + 24\sqrt{x} + 3^2 = 216 + 9 = 225.$$

Extracting the square root, $4\sqrt{x} + 3 = \pm 15$.

$$4\sqrt{x} = -3 \pm 15 = 12 \text{ or } -18.$$

Whence, $\sqrt{x} = 3 \text{ or } -\frac{9}{2}$.

Squaring, $x = 9 \text{ or } \frac{81}{4}$, *Ans.*

3. Solve the equation $16x^{-\frac{3}{2}} - 22x^{-\frac{3}{4}} = 3$.

Multiplying by 16, and adding 11^2 to both members,

$$16^2x^{-\frac{3}{2}} - 16 \times 22x^{-\frac{3}{4}} + 11^2 = 48 + 121 = 169.$$

Extracting the square root, $16x^{-\frac{3}{4}} - 11 = \pm 13$.

$$16x^{-\frac{3}{4}} = 11 \pm 13 = -2 \text{ or } 24.$$

Whence, $x^{-\frac{3}{4}} = -\frac{1}{8} \text{ or } \frac{3}{2}$.

Extracting the cube root, $x^{-\frac{1}{4}} = -\frac{1}{2} \text{ or } \left(\frac{3}{2}\right)^{\frac{1}{3}}$.

Raising to the fourth power, $x^{-1} = \frac{1}{16} \text{ or } \left(\frac{3}{2}\right)^{\frac{4}{3}}$.

Inverting both members, $x = 16 \text{ or } \left(\frac{2}{3}\right)^{\frac{4}{3}}$, *Ans.*

Note 2. In solving equations of the form $x^{\frac{p}{q}} = a$, first extract the root corresponding to the numerator of the fractional exponent, and then raise to the power corresponding to the denominator. Particular attention should be paid to the algebraic signs; see §§ 186 and 193.

EXAMPLES.

Solve the following equations:

4. $x^4 - 21x^2 = -108$.
5. $8x + 14\sqrt{x} = 15$.
6. $x^3 - 3x^{\frac{3}{2}} = 88$.
7. $32x^5 + \frac{1}{x^5} = -33$.
8. $12x^{-2} + x^{-1} = 35$.
9. $x^{\frac{6}{5}} + x^{\frac{3}{5}} = 702$.
10. $16x^{-6} - 30x^{-3} = 81$.
11. $7\sqrt[6]{x} - \frac{3}{\sqrt[6]{x}} = 20$.
12. $(2x^3 - 3)^2 = -26x^3 + 153$.
13. $(5x^{-2} - 2)^2 - 16(x^{-2} + 1)^2 = -76$.
14. $9x^{-\frac{4}{3}} - 22x^{-\frac{2}{3}} = -8$.
15. $3x^{\frac{4n}{3}} + 2x^{\frac{2n}{3}} = 16$.
16. $x^{-\frac{4}{3}} - 34x^{-\frac{2}{3}} = -225$.
17. $x^8 - 97x^4 + 1296 = 0$.
18. $3x^{\frac{5}{6}} - \frac{64}{x^{\frac{5}{6}}} = 94$.

19. $4x^{-\frac{3}{4}} + 27x^{-\frac{3}{8}} = 40.$

20. $8x^{-3} - 35x^{-\frac{3}{2}} = -27.$

21. $27\sqrt{x^3} + 10\sqrt[4]{x^3} = 128.$

22. $2x^{-5} + 59x^{-\frac{5}{2}} = 160.$

23. $\dfrac{x+b}{\sqrt{x}} = \dfrac{a+b}{\sqrt{a}}.$

24. $\sqrt{5+\sqrt{x}} + \sqrt{5-\sqrt{x}} = \dfrac{6}{\sqrt{5+\sqrt{x}}}.$

25. $\sqrt{3+\sqrt{x}} + \sqrt{4-\sqrt{x}} = \sqrt{7+2\sqrt{x}}.$

271. An equation may sometimes be solved with reference to an *expression*, by regarding it as a single quantity.

1. Solve the equation $(x-5)^3 - 3(x-5)^{\frac{3}{2}} = 40.$

Multiplying by 4, and adding 3^2 to both members,

$$4(x-5)^3 - 12(x-5)^{\frac{3}{2}} + 3^2 = 160 + 9 = 169.$$

Extracting the square root, $2(x-5)^{\frac{3}{2}} - 3 = \pm 13.$

$$2(x-5)^{\frac{3}{2}} = 3 \pm 13 = 16 \text{ or } -10.$$

Whence, $(x-5)^{\frac{3}{2}} = 8 \text{ or } -5.$

Extracting the cube root, $(x-5)^{\frac{1}{2}} = 2 \text{ or } -\sqrt[3]{5}.$

Squaring, $x - 5 = 4 \text{ or } \sqrt[3]{25}.$

Whence, $x = 9 \text{ or } 5 + \sqrt[3]{25}$, *Ans.*

Certain equations of the fourth degree may be solved by the rules for quadratics.

2. Solve the equation $x^4 + 12x^3 + 34x^2 - 12x - 35 = 0.$

The equation may be written

$$(x^4 + 12x^3 + 36x^2) - 2x^2 - 12x = 35.$$

That is, $(x^2 + 6x)^2 - 2(x^2 + 6x) = 35.$

Completing the square,

$$(x^2 + 6x)^2 - 2(x^2 + 6x) + 1 = 36.$$

Extracting the square root,

$$(x^2 + 6x) - 1 = \pm 6.$$

$$x^2 + 6x = 1 \pm 6 = 7 \text{ or } -5.$$

Completing the square, $x^2 + 6x + 9 = 16$ or 4.

Extracting the square root, $x + 3 = \pm 4$ or ± 2.

Whence, $x = -3 \pm 4$ or -3 ± 2

$= 1, -7, -1,$ or -5, *Ans.*

Note 1. In solving equations like the above, we first form a perfect square with the x^4 and x^3 terms, and a portion of the x^2 term. By § 258, the third term of the square is the square of the quotient obtained by dividing the x^3 term by twice the square root of the x^4 term.

3. Solve the equation $x^2 - 6x + 5\sqrt{x^2 - 6x + 20} = 46$.

Adding 20 to both members,

$$(x^2 - 6x + 20) + 5\sqrt{x^2 - 6x + 20} = 66.$$

Multiplying by 4, and adding 5^2 to both members,

$$4(x^2 - 6x + 20) + 20\sqrt{x^2 - 6x + 20} + 5^2 = 264 + 25 = 289.$$

Extracting the square root,

$$2\sqrt{x^2 - 6x + 20} + 5 = \pm 17.$$

$$2\sqrt{x^2 - 6x + 20} = -5 \pm 17 = 12 \text{ or } -22.$$

Whence, $\sqrt{x^2 - 6x + 20} = 6$ or -11.

Squaring, $x^2 - 6x + 20 = 36$ or 121.

Completing the square, $x^2 - 6x + 9 = 25$ or 110.

Extracting the square root, $x - 3 = \pm 5$ or $\pm\sqrt{110}$.

Whence, $x = 8, -2,$ or $3 \pm \sqrt{110}$, *Ans.*

Note 2. In solving equations like the above, add such a quantity to both members that the expression without the radical in the first member may be the same as that within, or some multiple of it.

EXAMPLES.

Solve the following equations:

4. $(x^2 - 2x)^2 - 18(x^2 - 2x) = -45$.

5. $x^4 + 8x^3 - 10x^2 - 104x + 105 = 0$.

6. $x^4 - 10x^3 + 23x^2 + 10x - 24 = 0$.

7. $x^2 + 7 + \sqrt{x^2 + 7} = 20.$

8. $\sqrt{3x - 2} - 5\sqrt[4]{3x - 2} = -6.$

9. $x^2 - 2x + 6\sqrt{x^2 - 2x + 5} = 11.$

10. $x^2 + 2x + 3 = \sqrt{x^2 + 2x + 9}.$

11. $\sqrt[3]{3x - 2x^2} - \sqrt[6]{3x - 2x^2} = 2.$

12. $(x^3 + 17)^{\frac{3}{2}} - 35(x^3 + 17)^{\frac{3}{4}} = -216.$

13. $(2x + 5)^{-5} + 31(2x + 5)^{-\frac{5}{2}} = 32.$

14. $x^4 + 14x^3 + 71x^2 + 154x + 120 = 0.$

15. $2x^2 - 3x + 6\sqrt{2x^2 - 3x + 2} = 14.$

16. $4x^4 - 12x^3 + 7x^2 + 3x - 2 = 0.$

17. $(3x^2 + x - 1)^3 - 26(3x^2 + x - 1)^{\frac{3}{2}} = 27.$

18. $4x^2 - 9x + 23 = 7\sqrt{4x^2 - 9x + 11}.$

19. $(x^2 + b^2)^2 = 2ax^3 + 2ab^2x - a^2x^2.$

20. $(x - a)^{\frac{4}{3}} - 5b(x - a)^{\frac{2}{3}} + 6b^2 = 0.$

21. $2(x^2 - 2x) + 3\sqrt{x^2 - 2x + 6} = 15.$

22. $3x^2 - 9x = 4\sqrt{x^2 - 3x + 5} - 11.$

23. $8(5x - 3)^{-\frac{4}{5}} - 6(5x - 3)^{-\frac{2}{5}} = -1.$

24. $2(2x^3 + 10)^{-\frac{1}{3}} + 3(2x^3 + 10)^{-\frac{1}{6}} = 2.$

25. $x^4 + 4ax^3 - 34a^2x^2 - 76a^3x + 105a^4 = 0.$

XXIV. SIMULTANEOUS EQUATIONS.

INVOLVING QUADRATICS.

272. An equation containing two unknown quantities is said to be *symmetrical* with respect to them when they can be interchanged without destroying the equality.

Thus, the equation $x^2 - xy + y^2 = 3$ is symmetrical with respect to x and y; for on interchanging x and y, it becomes $y^2 - yx + x^2 = 3$, which is equivalent to the first equation.

But the equation $x - y = 1$ is not symmetrical with respect to x and y; for on interchanging x and y, it becomes $y - x = 1$, or $x - y = -1$, which is a different equation.

273. An equation containing two unknown quantities is said to be *homogeneous* when the terms containing the unknown quantities are of the same degree with respect to them (§ 157).

Thus, the equation $x^2 - 3xy - 2y^2 = 1$ is homogeneous, for the terms containing x and y are of the second degree with respect to x and y.

But the equation $x^2 - 2y = 3$ is not homogeneous; for x^2 is of the second degree, and $2y$ of the first degree.

274. *On the use of the double signs* ± *and* ∓.

If two or more equations involve double signs, it will be understood that the equations can be read in two ways; first, reading all the *upper* signs together; second, reading all the *lower* signs together.

Thus, the equations $x = \pm 2$, $y = \pm 3$, can be read either

$$x = +2,\ y = +3, \text{ or } x = -2,\ y = -3.$$

Also, the equations $x = \pm 2$, $y = \mp 3$, can be read either

$$x = +2,\ y = -3, \text{ or } x = -2,\ y = +3.$$

275. Two equations of the second degree (§ 158) with two unknown quantities will generally produce, by elimination, an equation of the *fourth* degree with one unknown quantity; the rules already given are, therefore, not sufficient to solve all cases of simultaneous equations of the second degree with two unknown quantities.

Consider, for example, the equations

$$\begin{cases} x^2 + y = 5. & (1) \\ x + y^2 = 3. & (2) \end{cases}$$

From (1), $y = 5 - x^2.$

Substituting in (2), $x + 25 - 10x^2 + x^4 = 3;$

which is an equation of the fourth degree.

In several cases, however, the solution may be effected by means of the rules for quadratics.

276. Case I. *When each equation is in the form*

$$ax^2 + by^2 = c.$$

1. Solve the equations $\begin{cases} 3x^2 + 4y^2 = 76. & (1) \\ 3y^2 - 11x^2 = 4. & (2) \end{cases}$

Multiplying (1) by 3, $9x^2 + 12y^2 = 228.$

Multiplying (2) by 4, $\underline{12y^2 - 44x^2 = 16.}$

Subtracting, $53x^2 = 212.$

Then, $x^2 = 4$, and $x = \pm 2.$ (3)

Substituting from (3) in (1), $12 + 4y^2 = 76.$

$4y^2 = 64.$

Then, $y^2 = 16$, and $y = \pm 4.$

Ans. $x = 2,\ y = \pm 4$; or, $x = -2,\ y = \pm 4.$

Note. In this case there are four possible sets of values of x and y which satisfy the given equations:

1. $x = 2,\ y = 4.$
2. $x = 2,\ y = -4.$
3. $x = -2,\ y = 4.$
4. $x = -2,\ y = -4.$

It would be incorrect to leave the result in the form $x = \pm 2$, $y = \pm 4$; for, by § 274, this represents only the first and fourth of the above sets of values.

EXAMPLES.

Solve the following equations:

2. $\begin{cases} 4x^2 + y^2 = 61. \\ 2x^2 + 3y^2 = 93. \end{cases}$

4. $\begin{cases} 8x^2 - 11y^2 = 8. \\ 12x^2 + 13y^2 = 248. \end{cases}$

3. $\begin{cases} 5x^2 - 9y^2 = -121. \\ 7y^2 - 3x^2 = 105. \end{cases}$

5. $\begin{cases} x^2+y^2=5(a^2+b^2). \\ 4x^2-y^2=5a(3a-4b). \end{cases}$

277. Case II. *When one equation is of the second degree, and the other of the first.*

Equations of this kind may always be solved by finding the value of one of the unknown quantities in terms of the other from the simple equation, and substituting this value in the other equation.

1. Solve the equations $\begin{cases} 2x^2 - xy = 6y. & (1) \\ x + 2y = 7. & (2) \end{cases}$

From (2), $2y = 7 - x$, or $y = \frac{7-x}{2}$. (3)

Substituting in (1), $2x^2 - x\left(\frac{7-x}{2}\right) = 6\left(\frac{7-x}{2}\right)$.

Clearing of fractions, $4x^2 - 7x + x^2 = 42 - 6x$.

Or, $5x^2 - x = 42$.

Solving this equation, $x = 3$ or $-\frac{14}{5}$.

Substituting in (3), $y = \frac{7-3}{2}$ or $\frac{7+\frac{14}{5}}{2} = 2$ or $\frac{49}{10}$.

$$x = 3,\ y = 2;\ \text{or},\ x = -\frac{14}{5},\ y = \frac{49}{10},\ \textit{Ans.}$$

Note. Certain examples where one equation is of the *third* degree, and the other of the first, may be solved by the method of Case II.

EXAMPLES.

Solve the following equations:

2. $\begin{cases} 5x^2 - 3y^2 = -7. \\ 2x + y = 7. \end{cases}$

3. $\begin{cases} x + y = -3. \\ xy = -54. \end{cases}$

4. $\begin{cases} x-y=1. \\ x^2+y^2=113. \end{cases}$

5. $\begin{cases} x^2+xy-y^2=-19. \\ x-y=-7. \end{cases}$

6. $\begin{cases} x^3-y^3=-117. \\ x-y=-3. \end{cases}$

7. $\begin{cases} x^3+y^3=217. \\ x+y=7. \end{cases}$

8. $\begin{cases} x-y=1. \\ xy=a^2+a. \end{cases}$

9. $\begin{cases} \dfrac{x}{3}-\dfrac{y}{4}=-\dfrac{4}{3}. \\ \dfrac{6}{x}+\dfrac{4}{y}=1. \end{cases}$

10. $\begin{cases} x+y=2a. \\ x^2+y^2=2(a^2+b^2). \end{cases}$

11. $\begin{cases} x^3+27y^3=98. \\ x+3y=2. \end{cases}$

12. $\begin{cases} 2x^2-4xy+3y^2=11. \\ x-3y=5. \end{cases}$

13. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{2}. \\ 2x+2y=5. \end{cases}$

14. $\begin{cases} 7x^2+10xy=-8. \\ 5x+4y=-8. \end{cases}$

15. $\begin{cases} \dfrac{x}{y}+\dfrac{y}{x}=\dfrac{10}{3}. \\ 3x-2y=-12. \end{cases}$

278. CASE III. *When the equations are symmetrical with respect to x and y (§ 272), and one equation is of the second degree, and the other of the second or first.*

Equations of this kind may always be solved by combining them in such a way as to obtain the values of $x+y$ and $x-y$.

1. Solve the equations $\begin{cases} x+y=2. & (1) \\ xy=-15. & (2) \end{cases}$

Squaring (1), $x^2+2xy+y^2=4.$

Multiplying (2) by 4, $4xy \quad =-60.$

Subtracting, $x^2-2xy+y^2=\ 64.$

Extracting the square root, $x-y=\pm 8.$ (3)

Adding (1) and (3), $2x=2\pm 8=10$ or $-6.$

Whence, $x=5$ or $-3.$

Subtracting (3) from (1), $2y=2\mp 8=-6$ or $10.$

Whence, $y=-3$ or $5.$

$x=5,\ y=-3$; or, $x=-3,\ y=5,$ *Ans.*

Note 1. In subtracting ± 8 from 2, we have 2 ∓ 8, in accordance with the notation explained in § 274. In operating with double signs, $\pm$ is changed to $\mp$, and $\mp$ to $\pm$, whenever $+$ would be changed to $-$.

Note 2. The above equations may also be solved by the method of Case II.; but the symmetrical method is shorter and neater.

Certain examples in which one equation is of the *third* degree, and the other of the first or second, may be solved by the method of Case III.

2. Solve the equations
$$\begin{cases} x^3 - y^3 = 56. & (1) \\ x^2 + xy + y^2 = 28. & (2) \end{cases}$$

Dividing (1) by (2), $x - y = 2.$ (3)

Squaring (3), $x^2 - 2xy + y^2 = 4.$ (4)

Subtracting (4) from (2), $3xy = 24$, or $xy = 8.$ (5)

Adding (2) and (5), $x^2 + 2xy + y^2 = 36.$

Whence, $x + y = \pm 6.$ (6)

Adding (3) and (6), $2x = \pm 6 + 2 = 8$ or $-4.$

Whence, $x = 4$ or $-2.$

Subtracting (3) from (6), $2y = \pm 6 - 2 = 4$ or $-8.$

Whence, $y = 2$ or $-4.$

$x = 4, y = 2$; or, $x = -2, y = -4$. *Ans.*

Note 3. The above equations are not symmetrical according to the definition of § 272; but the method of Case III. may often be used in cases where the given equations are symmetrical except with respect to the *signs* of the terms.

3. Solve the equations
$$\begin{cases} x^2 + y^2 = 50. & (1) \\ xy = -7. & (2) \end{cases}$$

Multiplying (2) by 2, $2xy = -14.$ (3)

Adding (1) and (3), $x^2 + 2xy + y^2 = 36.$

Whence, $x + y = \pm 6.$ (4)

Subtracting (3) from (1), $x^2 - 2xy + y^2 = 64.$

Whence, $x - y = \pm 8.$ (5)

Adding (4) and (5), $2x = 6 \pm 8$, or $-6 \pm 8.$

Whence, $x = 7, -1, 1$, or $-7.$

Subtracting (5) from (4), $2y = 6 \mp 8$, or $-6 \mp 8.$

Whence, $y = -1, 7, -7$, or $1.$

$x = \pm 7, y = \mp 1$; or, $x = \pm 1, y = \mp 7$, *Ans.*

EXAMPLES.

Solve the following equations:

4. $\begin{cases} xy = 48. \\ x + y = 14. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 101. \\ x + y = -9. \end{cases}$

6. $\begin{cases} x^3 - y^3 = 37. \\ x^2 + xy + y^2 = 37. \end{cases}$

7. $\begin{cases} xy = 45. \\ x - y = -4. \end{cases}$

8. $\begin{cases} xy = 12. \\ x^2 + y^2 = 40. \end{cases}$

9.* $\begin{cases} x^3 - y^3 = 133. \\ x - y = 7. \end{cases}$

10. $\begin{cases} x^3 + y^3 = -217. \\ x + y = -7. \end{cases}$

11. $\begin{cases} x^2 + xy + y^2 = 39. \\ x + y = -2. \end{cases}$

12. $\begin{cases} x^2 + y^2 = 260. \\ x - y = -14. \end{cases}$

13. $\begin{cases} xy = -80. \\ x - y = 24. \end{cases}$

14. $\begin{cases} x^3 + y^3 = 504. \\ x^2 - xy + y^2 = 84. \end{cases}$

15. $\begin{cases} x^2 - xy + y^2 = 63. \\ x - y = -3. \end{cases}$

16. $\begin{cases} x^2 + y^2 = 305. \\ x - y = 21. \end{cases}$

17. $\begin{cases} x^2 + y^2 = 218. \\ xy = -91. \end{cases}$

18. $\begin{cases} x^3 + y^3 = -335. \\ x^2 - xy + y^2 = 67. \end{cases}$

19. $\begin{cases} xy = -150. \\ x - y = -31. \end{cases}$

279. CASE IV. *When each equation is of the second degree, and homogeneous* (§ 273).

Note 1. Certain equations which are of the second degree and homogeneous may be solved by the method of Case I. or Case III. (See Ex. 1, § 276, and Ex. 3, § 278.)

The method of Case IV. should be used only when the example cannot be solved by the methods of Cases I. or III.

1. Solve the equations $\begin{cases} x^2 - 2xy = 5. \\ x^2 + \quad y^2 = 29. \end{cases}$

Putting $y = vx$ in the given equations, we have

$$x^2 - 2vx^2 = 5\text{; or, } x^2 = \frac{5}{1 - 2v}; \tag{1}$$

and $$x^2 + v^2x^2 = 29\text{; or, } x^2 = \frac{29}{1 + v^2}.$$

* Divide the first equation by the second.

Equating the values of x^2, $\quad \dfrac{5}{1-2v}=\dfrac{29}{1+v^2}.$

Or, $\quad 5+5v^2=29-58v.$

Or, $\quad 5v^2+58v=24.$

Solving this equation, $\quad v=\dfrac{2}{5}$ or $-12.$

Substituting these values in (1), $\quad x^2=\dfrac{5}{1-\frac{4}{5}}$ or $\dfrac{5}{1+24}=25$ or $\dfrac{5}{25}$.

Whence, $\quad x=\pm 5$ or $\pm\dfrac{\sqrt{5}}{5}.$

Substituting the values of v and x in the equation $y=vx$,

If $v=\dfrac{2}{5}$ and $x=\pm 5$, $y=\dfrac{2}{5}(\pm 5)=\pm 2.$

If $v=-12$ and $x=\pm\dfrac{\sqrt{5}}{5}$, $y=-12\left(\pm\dfrac{\sqrt{5}}{5}\right)=\mp\dfrac{12\sqrt{5}}{5}.$

Ans. $x=\pm 5$, $y=\pm 2$; or, $x=\pm\frac{1}{5}\sqrt{5}$, $y=\mp\frac{12}{5}\sqrt{5}.$

Note 2. In finding y from the equation $y=vx$, care must be taken to multiply each pair of values of x by the *corresponding value* of v.

EXAMPLES.

Solve the following equations:

2. $\begin{cases} 2x^2-xy=28. \\ x^2+2y^2=18. \end{cases}$

3. $\begin{cases} x^2+xy=-6. \\ xy-y^2=-35. \end{cases}$

4. $\begin{cases} x^2+xy+y^2=63. \\ x^2-y^2=-27. \end{cases}$

5. $\begin{cases} x^2+3y^2=28. \\ x^2+xy+2y^2=16. \end{cases}$

6. $\begin{cases} x^2-2xy=84. \\ 2xy-y^2=-64. \end{cases}$

7. $\begin{cases} 3x^2+xy-3y^2=33. \\ 2x^2-y^2=23. \end{cases}$

8. $\begin{cases} x^2+5xy-y^2=-7. \\ x^2+3xy-2y^2=-4. \end{cases}$

9. $\begin{cases} x^2-xy-12y^2=8. \\ x^2+xy-10y^2=20. \end{cases}$

10. $\begin{cases} 5x^2-4xy=33. \\ 27x^2-32xy-4y^2=55. \end{cases}$

11. $\begin{cases} 3x^2+xy+y^2=47. \\ 4x^2-3xy-y^2=-39. \end{cases}$

MISCELLANEOUS AND REVIEW EXAMPLES.

280. No general rules can be given for the solution of examples which do not come under the cases just considered. Various artifices are employed, familiarity with which can only be gained by experience.

1. Solve the equations $\begin{cases} x^3 - y^3 = 19. & (1) \\ x^2y - xy^2 = 6. & (2) \end{cases}$

Multiplying (2) by 3, $3x^2y - 3xy^2 = 18.$ (3)

Subtracting (3) from (1), $x^3 - 3x^2y + 3xy^2 - y^3 = 1.$

Extracting the cube root, $x - y = 1.$ (4)

Dividing (2) by (4), $xy = 6.$ (5)

Solving equations (4) and (5) by the method of Case III., we find

$x = 3,\ y = 2$; or, $x = -2,\ y = -3$, *Ans.*

2. Solve the equations $\begin{cases} x^3 + y^3 = 9xy. \\ x + y = 6. \end{cases}$

Putting $x = u + v$ and $y = u - v$, we have

$(u+v)^3 + (u-v)^3 = 9(u+v)(u-v),$ (1)

and $(u+v) + (u-v) = 6.$ (2)

Reducing (1), $2u^3 + 6uv^2 = 9(u^2 - v^2).$ (3)

Reducing (2), $2u = 6$, or $u = 3$.

Substituting the value of u in (3), $54 + 18v^2 = 9(9 - v^2).$

Whence, $v^2 = 1$, or $v = \pm 1$.

Therefore, $x = u + v = 3 \pm 1 = 4$ or 2,

and $y = u - v = 3 \mp 1 = 2$ or 4.

$x = 4,\ y = 2$; or, $x = 2,\ y = 4$, *Ans.*

Note. The artifice of substituting $u + v$ and $u - v$ for x and y is applicable in any case where the given equations are *symmetrical* with respect to x and y (§ 272). See also Ex. 4, p. 256.

3. Solve the equations $\begin{cases} x^2 + y^2 + 2x + 2y = 23. & (1) \\ xy = 6. & (2) \end{cases}$

Multiplying (2) by 2, $2xy = 12.$ (3)

Adding (1) and (3), $x^2 + 2xy + y^2 + 2x + 2y = 35.$

Or, $(x+y)^2+2(x+y)=35.$

Completing the square, $(x+y)^2+2(x+y)+1=36.$

Whence, $(x+y)+1=\pm 6,$

or $x+y=5 \text{ or } -7.$ (4)

Squaring (4), $x^2+2xy+y^2=25 \text{ or } 49.$

Multiplying (2) by 4, $4xy \quad = 24.$

Subtracting, $x^2-2xy+y^2=1 \text{ or } 25.$

Whence, $x-y=\pm 1 \text{ or } \pm 5.$ (5)

Adding (4) and (5), $2x=5\pm 1, \text{ or } -7\pm 5.$

Whence, $x=3,\ 2,\ -1, \text{ or } -6.$

Subtracting (5) from (4), $2y=5\mp 1, \text{ or } -7\mp 5.$

Whence, $y=2,\ 3,\ -6, \text{ or } -1.$

$x=3,\ y=2;\ x=2,\ y=3;\ x=-1,\ y=-6;$ or, $x=-6,\ y=-1,$ *Ans.*

4. Solve the equations $\begin{cases} x^4+y^4=97. \\ x+y=-1. \end{cases}$

Putting $x=u+v$ and $y=u-v$, we have

$$(u+v)^4+(u-v)^4=97, \qquad (1)$$

and

$$(u+v)+(u-v)=-1. \qquad (2)$$

Reducing (1), $2u^4+12u^2v^2+2v^4=97.$ (3)

Reducing (2), $2u=-1, \text{ or } u=-\frac{1}{2}.$

Substituting in (3), $\frac{1}{8}+3v^2+2v^4=97.$

Solving this equation, $v^2=\frac{25}{4} \text{ or } -\frac{31}{4}.$

Whence, $v=\pm\frac{5}{2} \text{ or } \pm\frac{\sqrt{-31}}{2}.$

Then,

$$x=u+v=-\frac{1}{2}\pm\frac{5}{2} \text{ or } -\frac{1}{2}\pm\frac{\sqrt{-31}}{2}=2,\ -3, \text{ or } \frac{-1\pm\sqrt{-31}}{2},$$

and

$$y=u-v=-\frac{1}{2}\mp\frac{5}{2} \text{ or } -\frac{1}{2}\mp\frac{\sqrt{-31}}{2}=-3,\ 2, \text{ or } \frac{-1\mp\sqrt{-31}}{2}.$$

$x=2,\ y=-3;\ x=-3,\ y=2;$ or, $x=\frac{-1\pm\sqrt{-31}}{2},\ y=\frac{-1\mp\sqrt{-31}}{2},$

Ans.

EXAMPLES.

Solve the following equations:

5. $\begin{cases} x^2+y^2=1. \\ xy=-\dfrac{12}{25}. \end{cases}$

6. $\begin{cases} x^2+y^2+x-y=26. \\ xy=12. \end{cases}$

7. $\begin{cases} 2x^2-3xy=-4. \\ 4xy-5y^2=3. \end{cases}$

8. $\begin{cases} 4x^2-5xy=19. \\ xy+y^2=6. \end{cases}$

9. $\begin{cases} \dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{2}. \\ \dfrac{1}{xy}=-\dfrac{1}{18}. \end{cases}$

10. $\begin{cases} x^2+2y^2=47+2x. \\ x^2-2y^2=-7. \end{cases}$

11. $\begin{cases} x^2+xy+y^2=97. \\ x-y=19. \end{cases}$

12. $\begin{cases} x^3+y^3=756. \\ x^2-xy+y^2=63. \end{cases}$

13. $\begin{cases} x^2y^2+28xy-480=0. \\ 2x+y=11. \end{cases}$

14. $\begin{cases} \dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{5}{16}. \\ \dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{4}. \end{cases}$

15. $\begin{cases} x^4+y^4=17. \\ x+y=1. \end{cases}$

16. $\begin{cases} x^2-y^2=3. \\ xy=-2. \end{cases}$

17. $\begin{cases} x^2+4y^2+3x=22 \\ 2xy+3y+9=0. \end{cases}$

18. $\begin{cases} 3x^2-5xy+2y^2=-3. \\ 4x-5y=10. \end{cases}$

19. $\begin{cases} xy=a^2-1. \\ x+y=2a. \end{cases}$

20. $\begin{cases} \dfrac{1}{x^3}+\dfrac{1}{y^3}=91. \\ \dfrac{1}{x}+\dfrac{1}{y}=7. \end{cases}$

21. $\begin{cases} \dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{10}{3}. \\ x^2+y^2=45. \end{cases}$

22. $\begin{cases} x^3+y^3=2a^3+6ab^2. \\ xy(x+y)=2a^3-2ab^2. \end{cases}$

23. $\begin{cases} 2x^2-3xy=15a-10a^2. \\ 3x+2y=12a-13. \end{cases}$

24.* $\begin{cases} x^4+x^2y^2+y^4=91. \\ x^2+xy+y^2=13. \end{cases}$

25. $\begin{cases} x^2+y^2=13(a^2+1). \\ x+y=5a-1. \end{cases}$

26. $\begin{cases} 2x^2+3xy-4y^2=-20. \\ 5x^2-7y^2=-8. \end{cases}$

* Divide the first equation by the second.

27. $\begin{cases} x^2 + xy + y^2 = 7a^2 - 13ab + 7b^2. \\ x^2 - xy + y^2 = 3a^2 - 3ab + 3b^2. \end{cases}$

28. $\begin{cases} x^4 + y^4 = 97. \\ x - y = 5. \end{cases}$

29. $\begin{cases} 9x^2 - xy - y = 51. \\ -5xy + y^2 + 3x = 81. \end{cases}$

30. $\begin{cases} x - y = 19. \\ \sqrt[3]{x} - \sqrt[3]{y} = 1. \end{cases}$

31. $\begin{cases} \dfrac{x^2}{y} + \dfrac{y^2}{x} = -\dfrac{7}{2}. \\ x + y = 1. \end{cases}$

32. $\begin{cases} x^3 - y^3 = 3a^2 + 3a + 1. \\ x - y = 1. \end{cases}$

33. $\begin{cases} x^2y - x = -14. \\ x^4y^2 + x^2 = 148. \end{cases}$

34. $\begin{cases} x^2 - xy = 27y. \\ xy - y^2 = 3x. \end{cases}$

35. $\begin{cases} \dfrac{x+y}{x-y} + \dfrac{2x-y}{x+2y} = \dfrac{15}{4}. \\ x - 3y = -2. \end{cases}$

36. $\begin{cases} y(x-a) = 2ab. \\ x(y-b) = 2ab. \end{cases}$

37. $\begin{cases} x^5 - y^5 = 31. \\ x - y = 1. \end{cases}$

38. $\begin{cases} x^2 = x + y. \\ y^2 = 3y - x. \end{cases}$

39. $\begin{cases} \sqrt{x^2 + 7} = 6 - y. \\ \sqrt{x^4 + 22y^2} = 22 - x^2. \end{cases}$

40. $\begin{cases} 53x^2 - 128xy + 64y^2 = 5. \\ 26x^2 - 62xy + 32y^2 = 5. \end{cases}$

41. $\begin{cases} xy - (x-y) = 1. \\ x^2y^2 + (x-y)^2 = 13. \end{cases}$

42. $\begin{cases} xy + (x-y) = -5. \\ xy(x-y) = -84. \end{cases}$

43. $\begin{cases} x^2 - xy + y^2 = 12. \\ x^3 + y^3 + 3xy = -48. \end{cases}$

44. $\begin{cases} x^2 + xy + y^2 = 7. \\ x + y = 5 + xy. \end{cases}$

45. $\begin{cases} 2x^2 + 2y^2 = 5xy. \\ x^5 + y^5 = 33. \end{cases}$

46. $\begin{cases} x^2 - 2xy + 3xz = -16. \\ 2x - 3y = 7. \\ 3x + 5z = -14. \end{cases}$

PROBLEMS.

Note. In the following problems, as in those of § 268, only those answers are to be retained which satisfy the given conditions.

281. 1. The sum of the squares of two numbers is 52, and their difference is one-fifth of their sum. Find the numbers.

2. The difference of the squares of two numbers is 16, and their product is 15. Find the numbers.

3. If the length of a rectangular field were increased by 2 rods, and its width diminished by 5 rods, its area would be 80 square rods; and if its length were diminished by 4 rods, and its width increased by 3 rods, its area would be 168 square rods. Find its length and width.

4. The difference of the cubes of two numbers is 218, and the sum of their squares is equal to 109 minus their product. Find the numbers.

5. If the product of two numbers be multiplied by their sum, the result is 70; and the sum of the cubes of the numbers is 133. Find the numbers.

6. A farmer bought 4 cows and 8 sheep for $600. He bought 5 more cows for $490 than sheep for $80. Find the price of each.

7. Find a number of two figures such that, if its digits be inverted, the difference of the number thus formed and the original number is 9, and their product 736.

8. The sum of two numbers exceeds the product of their square roots by 7; and if the product of the numbers be added to the sum of their squares, the result is 133. Find the numbers.

9. The sum of the terms of a fraction is 13. If the numerator be decreased by 2, and the denominator increased by 2, the product of the resulting fraction and the original fraction is $\frac{8}{15}$. Find the fraction.

10. A rectangular mirror is surrounded by a frame $3\frac{1}{2}$ inches wide. The area of the mirror is 384 square inches, and of the frame 329 square inches. Find the length and width of the mirror.

11. A crew row up stream 18 miles in 4 hours more time than it takes them to return. If they row at two-thirds their usual rate, their rate up stream would be one mile an hour. Find their rate in still water, and the rate of the stream.

12. A rectangular field contains $2\frac{1}{4}$ acres. If its length were decreased by 10 rods, and its width by 2 rods, its area would be less by an acre. Find its length and width.

13. A distributes $180 equally among a certain number of persons. B distributes the same sum between a number of people less by 40, and gives to each $6 more than A does. How many persons are there, and how much does A give to each?

14. A, B, and C together can do a piece of work in one hour. B does twice as much work as A in a given time; and B alone requires one hour more than C alone to perform the work. In what time could each alone do the work?

15. If the length of a rectangular field were increased by one-eighth of itself, and its width decreased by one-sixth of itself, its area would be decreased by 60 square rods, and its perimeter by 2 rods. Find its length and width.

16. If the product of two numbers be added to their difference, the result is 26; and the sum of their squares exceeds their difference by 50. Find the numbers.

(Represent the numbers by $x + y$ and $x - y$.)

17. A sets out to walk to a town 21 miles off, and one hour afterwards B starts to follow him. When B has overtaken A, he turns back, and reaches the starting-point at the same instant that A reaches his destination. B walked at the rate of 4 miles an hour. Find A's rate, and the distance from the starting-point to where B overtook A.

18. A tank can be filled by three pipes, A, B, and C, when opened together, in $2\frac{2}{11}$ hours. If A filled at the same rate as B, it would take 3 hours for A, B, and C to fill the tank; and the sum of the times required by A and C alone to fill the tank is double the time required by B alone. In what time can each pipe alone fill the tank?

19. The sum of two numbers is 4, and the sum of their fifth powers is 244. Find the numbers.

XXV. THEORY OF QUADRATIC EQUATIONS.

282. Sum and Product of the Roots.

Let r_1 and r_2 denote the roots of the equation $x^2 + px = q$.

By § 265, $r_1 = \frac{-p + \sqrt{p^2 + 4q}}{2}$, and $r_2 = \frac{-p - \sqrt{p^2 + 4q}}{2}$.

Adding these values, $r_1 + r_2 = \frac{-2p}{2} = -p$.

Multiplying them together, we have

$$r_1 r_2 = \frac{p^2 - (p^2 + 4q)}{4} \text{ (§ 80)} = \frac{-4q}{4} = -q.$$

Hence, *if a quadratic equation is in the form* $x^2 + px = q$, *the sum of the roots is equal to the coefficient of* x *with its sign changed, and the product of the roots is equal to the second member with its sign changed.*

1. Find the sum and product of the roots of the equation $2x^2 - 7x - 15 = 0$.

Transposing -15, and dividing by 2, the equation becomes

$$x^2 - \frac{7x}{2} = \frac{15}{2}.$$

Hence, the sum of the roots is $\frac{7}{2}$, and their product $-\frac{15}{2}$.

EXAMPLES.

Find by inspection the sum and product of the roots of:

2. $x^2 + 7x + 6 = 0$.
3. $x^2 - x + 12 = 0$.
4. $x^2 + 3x - 1 = 0$.
5. $3x^2 - x - 6 = 0$.
6. $12x^2 - 4x + 3 = 0$.
7. $9x - 21x^2 + 7 = 0$.
8. $4 - x - 6x^2 = 0$.
9. $14x^2 + 8ax + 21a^2 = 0$.

283. Formation of Equations.

By aid of the principles of § 282, a quadratic equation may be formed which shall have any required roots.

For, let r_1 and r_2 denote the roots of the equation

$$x^2+px-q=0. \qquad (1)$$

Then by § 282, $p=-r_1-r_2$, and $-q=r_1r_2$.

Substituting these values in (1), we have

$$x^2-r_1x-r_2x+r_1r_2=0.$$

That is, $(x-r_1)(x-r_2)=0.$ (§ 93)

Hence, any quadratic equation can be written in the form

$$(x-r_1)(x-r_2)=0, \qquad (2)$$

where r_1 and r_2 are its roots.

Therefore, to form a quadratic equation which shall have any required roots,

Subtract each of the roots from x, and place the product of the resulting expressions equal to zero.

1. Form the quadratic equation whose roots shall be 4 and $-\frac{7}{4}$.

By the rule, $(x-4)\left(x+\frac{7}{4}\right)=0.$

Multiplying by 4, $(x-4)(4x+7)=0.$

Whence, $4x^2-9x-28=0,$ *Ans.*

EXAMPLES.

Form the quadratic equations whose roots shall be:

2. 6, 9.
3. 2, -3.
4. 1, $-\frac{2}{3}$.
5. -4, $-\frac{11}{2}$.
6. $\frac{1}{5}$, $\frac{5}{6}$.
7. $-\frac{5}{7}$, $\frac{3}{4}$.
8. $-\frac{17}{8}$, 0.
9. $-\frac{5}{4}$, $-\frac{8}{9}$.

10. $2a+b, a-3b$. 12. $3+7\sqrt{2}, 3-7\sqrt{2}$.

11. $a+3m, a-3m$. 13. $\frac{1}{2}(-\sqrt{a}+\sqrt{b}), \frac{1}{2}(-\sqrt{a}-\sqrt{b})$.

FACTORING.

284. Factoring of Quadratic Expressions.

A *quadratic expression* is an expression of the form

$$ax^2+bx+c.$$

The principles of § 283 serve to resolve such an expression into two factors, each of the first degree in x.

We have, $$ax^2+bx+c=a\left(x^2+\frac{bx}{a}+\frac{c}{a}\right). \qquad (1)$$

Now let r_1 and r_2 denote the roots of the equation

$$x^2+\frac{bx}{a}+\frac{c}{a}=0.$$

By § 283, (2), the equation can be written in the form

$$(x-r_1)(x-r_2)=0.$$

Hence, the *expression* $x^2+\frac{bx}{a}+\frac{c}{a}$ can be written

$$(x-r_1)(x-r_2).$$

Substituting in (1), we have

$$ax^2+bx+c=a(x-r_1)(x-r_2).$$

But r_1 and r_2 are the roots of the equation $x^2+\frac{bx}{a}+\frac{c}{a}=0$, or $ax^2+bx+c=0$; which, we observe, is obtained by *placing the given expression equal to zero.*

We then have the following rule:

To factor a quadratic expression, place it equal to zero, and solve the equation thus formed.

Then the required factors are the coefficient of x^2 in the given expression, x minus the first root, and x minus the second root.

EXAMPLES.

285. 1. Factor $6x^2+7x-3$.

Solving the equation $6x^2+7x-3=0$, we have by § 265,

$$x=\frac{-7\pm\sqrt{49+72}}{12}=\frac{-7\pm 11}{12}=\frac{1}{3}\text{ or }-\frac{3}{2}.$$

Then by the rule, $6x^2+7x-3=6\left(x-\frac{1}{3}\right)\left(x+\frac{3}{2}\right)$

$$=3\left(x-\frac{1}{3}\right)\times 2\left(x+\frac{3}{2}\right)$$

$$=(3x-1)(2x+3),\ \textit{Ans.}$$

2. Factor $4+13x-12x^2$.

Solving the equation $4+13x-12x^2=0$, we have by § 265,

$$x=\frac{-13\pm\sqrt{169+192}}{-24}=\frac{-13\pm 19}{-24}=-\frac{1}{4}\text{ or }\frac{4}{3}.$$

Whence, $4+13x-12x^2=-12\left(x+\frac{1}{4}\right)\left(x-\frac{4}{3}\right)$

$$=4\left(x+\frac{1}{4}\right)\times(-3)\left(x-\frac{4}{3}\right)$$

$$=(1+4x)(4-3x),\ \textit{Ans.}$$

Factor the following:

3. $x^2-13x+42$.
4. $x^2+15x+44$.
5. $x^2-9x-36$.
6. $3x^2+7x-6$.
7. $5x^2+18x+16$.
8. $6x^2-11x+3$.
9. $15x^2-14x-8$.
10. $20-7x-3x^2$.
11. $35-11x-6x^2$.
12. $12+28x-5x^2$.
13. $3x^2-17ax-28a^2$.
14. $6x^2-23mx+21m^2$.
15. $14x^2+25x+6$.
16. $18x^2-15x+2$.
17. $5-19x-4x^2$.
18. $18x^2+31x+6$.
19. $45+7x-12x^2$.
20. $42+23x-10x^2$.
21. $24x^2-26x+5$.
22. $8x^2+38x+35$.
23. $21x^2-10xy-24y^2$.
24. $7x^2+37abx-30a^2b^2$.

25. Factor $2x^2 - 3xy - 2y^2 - 7x + 4y + 6$.

Placing the expression equal to zero, we have

$$2x^2 - 3xy - 2y^2 - 7x + 4y + 6 = 0,$$

or

$$2x^2 - (3y + 7)x = 2y^2 - 4y - 6.$$

Solving this by the formula of § 265,

$$x = \frac{3y + 7 \pm \sqrt{(3y + 7)^2 + 16y^2 - 32y - 48}}{4}$$

$$= \frac{3y + 7 \pm \sqrt{25y^2 + 10y + 1}}{4} = \frac{3y + 7 \pm (5y + 1)}{4}$$

$$= \frac{8y + 8}{4} \text{ or } \frac{-2y + 6}{4} = 2y + 2 \text{ or } \frac{-y + 3}{2}.$$

Therefore,

$$2x^2 - 3xy - 2y^2 - 7x + 4y + 6 = 2[x - (2y + 2)]\left[x - \frac{-y + 3}{2}\right]$$

$$= (x - 2y - 2)(2x + y - 3), \textit{ Ans.}$$

Factor the following:

26. $x^2 + xy - 12y^2 + 7x + 7y + 12$.

27. $x^2 - xy - 2y^2 + x - 5y - 2$.

28. $x^2 - 4y^2 + 3x + 10y - 4$.

29. $2x^2 + 7xy - 4y^2 + x + 13y - 3$.

30. $3a^2 - 5ab - 2b^2 - 7a + 2$.

31. $6 - 15y - 5x + 9y^2 + 9xy - 4x^2$.

32. $6x^2 - 9xy + xz - 15y^2 - 13yz - 2z^2$.

286. If the coefficient of x^2 is a perfect square, it is convenient to factor the expression by the artifice of completing the square (§ 260) in connection with § 99.

1. Factor $9x^2 - 9x - 4$.

By § 260, the expression $9x^2 - 9x$ will become a perfect square by adding to it the square of $\frac{9}{2\sqrt{9}}$, or $\frac{3}{2}$. Then,

$$9x^2 - 9x - 4 = 9x^2 - 9x + \left(\frac{3}{2}\right)^2 - \frac{9}{4} - 4 = \left(3x - \frac{3}{2}\right)^2 - \frac{25}{4}$$

Factoring as in § 99, we have

$$9x^2 - 9x - 4 = \left(3x - \frac{3}{2} + \frac{5}{2}\right)\left(3x - \frac{3}{2} - \frac{5}{2}\right)$$
$$= (3x + 1)(3x - 4), \textit{ Ans.}$$

If the x^2 term is negative, the entire expression should be enclosed in a parenthesis preceded by a $-$ sign.

2. Factor $3 - 12x - 4x^2$.

$$\begin{aligned} 3 - 12x - 4x^2 &= -(4x^2 + 12x - 3) \\ &= -(4x^2 + 12x + 3^2 - 9 - 3) \\ &= -[(2x + 3)^2 - 12] \\ &= (2x + 3 + \sqrt{12}) \times (-1)(2x + 3 - \sqrt{12}) \\ &= (2\sqrt{3} + 3 + 2x)(2\sqrt{3} - 3 - 2x), \textit{ Ans.} \end{aligned}$$

EXAMPLES.

Factor the following:

3. $x^2 - 5x + 4$.
4. $4x^2 + 16x + 15$.
5. $9x^2 - 18x + 8$.
6. $16x^2 + 16x - 21$.
7. $x^2 + 2x - 11$.
8. $4x^2 + 4x - 1$.
9. $36x^2 + 24x - 5$.
10. $4x^2 + 5x - 6$.
11. $25x^2 + 30x + 6$.
12. $4 + 12x - 9x^2$.
13. $49x^2 + 56x + 12$.
14. $5 + 38x - 16x^2$.

287. Certain trinomials of the form $ax^4 + bx^2 + c$, where a and c are perfect squares, may be resolved into two factors by the artifice of completing the square.

1. Factor $9x^4 - 28x^2 + 4$.

By § 96, the expression will become a perfect square if its middle term is $-12x^2$.

Thus,
$$\begin{aligned} 9x^4 - 28x^2 + 4 &= (9x^4 - 12x^2 + 4) - 16x^2 \\ &= (3x^2 - 2)^2 - (4x)^2 \\ &= (3x^2 - 2 + 4x)(3x^2 - 2 - 4x) \qquad (\S\ 99) \\ &= (3x^2 + 4x - 2)(3x^2 - 4x - 2), \textit{ Ans.} \end{aligned}$$

2. Factor $a^4 + a^2b^2 + b^4$.

$$\begin{aligned} a^4 + a^2b^2 + b^4 &= (a^4 + 2a^2b^2 + b^4) - a^2b^2 \\ &= (a^2 + b^2)^2 - a^2b^2 \\ &= (a^2 + b^2 + ab)(a^2 + b^2 - ab) \\ &= (a^2 + ab + b^2)(a^2 - ab + b^2), \textit{ Ans.} \end{aligned}$$

3. Factor $x^4 + 1$.

$$\begin{aligned} x^4 + 1 &= (x^4 + 2x^2 + 1) - 2x^2 \\ &= (x^2 + 1)^2 - (x\sqrt{2})^2 \\ &= (x^2 + x\sqrt{2} + 1)(x^2 - x\sqrt{2} + 1), \textit{ Ans.} \end{aligned}$$

EXAMPLES.

Factor the following:

4. $x^4 + 2x^2 + 9$.

5. $x^4 - 19x^2 + 25$.

6. $4a^4 + 7a^2b^2 + 16b^4$.

7. $9x^4 - 28x^2y^2 + 4y^4$.

8. $16m^4 - m^2n^2 + n^4$.

9. $4a^4 - 53a^2 + 49$.

10. $9x^4 + 5x^2 + 9$.

11. $4m^4 - 13m^2 + 4$.

12. $x^4 + 16$.

13. $x^4 - 5x^2 + 1$.

14. $9a^4 - 55a^2x^2 + 25x^4$.

15. $16a^4 + 47a^2m^2 + 36m^4$.

16. $25x^4 - 21x^2 + 4$.

17. $25m^4 + 36m^2x^2 + 16x^4$.

18. $16x^4 - 60x^2y^2 + 49y^4$.

19. $36a^4 - 68a^2b^2 + 25b^4$.

288. Certain equations of the fourth degree may be solved by factoring the first member by the method of § 287, and then proceeding as in § 267.

1. Solve the equation $x^4 + 1 = 0$.

By Ex. 3, § 287, the equation may be written

$$(x^2 + x\sqrt{2} + 1)(x^2 - x\sqrt{2} + 1) = 0.$$

Then, as in § 267, $x^2 + x\sqrt{2} + 1 = 0$, and $x^2 - x\sqrt{2} + 1 = 0$.

Solving the equation $x^2 + x\sqrt{2} + 1 = 0$, we have by § 265,

$$x = \frac{-\sqrt{2} \pm \sqrt{2 - 4}}{2} = \frac{-\sqrt{2} \pm \sqrt{-2}}{2}$$

Solving the equation $x^2 - x\sqrt{2} + 1 = 0$, we have

$$x = \frac{\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{\sqrt{2} \pm \sqrt{-2}}{2}.$$

EXAMPLES.

Solve the following:

2. $x^4 - 2x^2 + 25 = 0$.
3. $x^4 - 18x^2 + 9 = 0$.
4. $4x^4 - 5x^2 + 1 = 0$.
5. $x^4 + x^2 + 1 = 0$.
6. $x^4 - 9x^2 + 9 = 0$.
7. $x^4 + 81 = 0$.

DISCUSSION OF THE GENERAL EQUATION.

289. By § 265, the roots of the equation $x^2 + px = q$ are

$$r_1 = \frac{-p + \sqrt{p^2 + 4q}}{2}, \text{ and } r_2 = \frac{-p - \sqrt{p^2 + 4q}}{2}.$$

We will now discuss these values for all possible real values of p and q.

I. *Suppose q positive.*

Since p^2 is essentially positive (§ 186), the expression under the radical sign is positive, and greater than p^2.

Therefore, the radical is numerically greater than p.

Hence, r_1 is positive, and r_2 is negative.

If p is positive, r_2 is numerically greater than r_1; that is, the negative root is numerically the greater.

If p is zero, the roots are numerically equal.

If p is negative, r_1 is numerically greater than r_2; that is, the positive root is numerically the greater.

II. *Suppose* $q = 0$.

The expression under the radical sign is now equal to p^2.

Therefore, the radical is numerically equal to p.

If p is positive, r_1 is zero, and r_2 is negative.

If p is negative, r_1 is positive, and r_2 is zero.

III. *Suppose q negative, and $4q$ numerically $< p^2$.*

The expression under the radical sign is now positive, and less than p^2.

Therefore, the radical is numerically less than p.

If p is positive, both roots are negative.

If p is negative, both roots are positive.

IV. *Suppose q negative, and $4q$ numerically equal to p^2.*

The expression under the radical sign now equals zero.

Hence, r_1 is equal to r_2.

If p is positive, both roots are negative.

If p is negative, both roots are positive.

V. *Suppose q negative, and $4q$ numerically $> p^2$.*

The expression under the radical sign is now negative.

Hence, both roots are imaginary (§ 248).

The roots are both *rational* or both *irrational*, according as $p^2 + 4q$ *is* or *is not* a perfect square.

EXAMPLES.

290. 1. Determine by inspection the nature of the roots of the equation $2x^2 - 5x - 18 = 0$.

The equation may be written $x^2 - \frac{5x}{2} = 9$; here $p = -\frac{5}{2}$ and $q = 9$.

Since q is positive and p negative, the roots are one positive and the other negative; and the positive root is numerically the greater.

In this case, $p^2 + 4q = \frac{25}{4} + 36 = \frac{169}{4}$; a perfect square.

Hence, the roots are both rational.

Determine by inspection the nature of the roots of the following:

2. $6x^2 + 7x - 5 = 0$.

3. $10x^2 + 17x + 3 = 0$.

4. $4x^2 - x = 0$.

5. $4x^2 - 20x + 25 = 0$.

6. $x^2 - 21x + 200 = 0$.

7. $16x^2 - 9 = 0$.

8. $9x^2 - 1 = 12x$.

9. $25x^2 + 30x + 9 = 0$.

10. $7x^2 + 3x = 0$.

11. $41x = 20x^2 + 20$.

XXVI. ZERO AND INFINITY.

VARIABLES AND LIMITS.

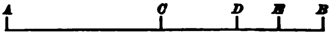
291. A *variable quantity*, or simply a *variable*, is a quantity which may assume, under the conditions imposed upon it, an indefinitely great number of different values.

A *constant* is a quantity which remains unchanged throughout the same discussion.

292. A *limit* of a variable is a constant quantity, the difference between which and the variable may be made less than any assigned quantity, however small, but cannot be made equal to zero.

In other words, a limit of a variable is a fixed quantity to which the variable approaches indefinitely near, but never actually reaches.

Suppose, for example, that a point moves from A towards B under the condition that it shall move, during successive equal intervals of time, first from A to C, half-way between A and B; then to D, half-way between C and B; then to E, half-way between D and B; and so on indefinitely.



In this case, the distance between the moving point and B can be made less than any assigned quantity, however small, but cannot be made equal to zero.

Hence, the distance from A to the moving point is a variable which approaches the constant value AB as a limit.

Again, the distance from the moving point to B is a variable which approaches the limit 0.

293. A problem is said to be *indeterminate* when the number of solutions is indefinitely great. (Compare § 159.)

294. Interpretation of $\frac{a}{0}$.

Consider the series of fractions

$$\frac{a}{3}, \frac{a}{.3}, \frac{a}{.03}, \frac{a}{.003}, \cdots;$$

where each denominator after the first is one-tenth of the preceding denominator.

It is evident that, by sufficiently continuing the series, the denominator may be made less than any assigned quantity, however small, and the value of the fraction greater than any assigned quantity, however great.

In other words,

If the numerator of a fraction remains constant, while the denominator approaches the limit 0, *the value of the fraction increases without limit.*

It is customary to express this principle as follows:

$$\frac{a}{0} = \infty.$$

Note. The symbol ∞ is called *Infinity.*

295. Interpretation of $\frac{a}{\infty}$.

Consider the series of fractions

$$\frac{a}{3}, \frac{a}{30}, \frac{a}{300}, \frac{a}{3000}, \cdots;$$

where each denominator after the first is ten times the preceding denominator.

It is evident that, by sufficiently continuing the series, the denominator may be made greater than any assigned quantity, however great, and the value of the fraction less than any assigned quantity, however small.

In other words,

If the numerator of a fraction remains constant, while the denominator increases without limit, the value of the fraction approaches the limit 0.

It is customary to express this principle as follows:

$$\frac{a}{\infty} = 0.$$

296. It must be clearly understood that no *literal meaning* can be attached to such results as

$$\frac{a}{0} = \infty, \text{ or } \frac{a}{\infty} = 0;$$

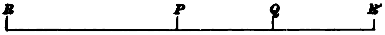
for there can be no such thing as division unless the divisor is a *finite quantity*.

If such forms occur in mathematical investigations, they must be interpreted as indicated in §§ 294 and 295. (Compare note to § 395.)

THE PROBLEM OF THE COURIERS.

297. The discussion of the following problem will serve to further illustrate the form $\frac{a}{0}$, besides furnishing an interpretation of the form $\frac{0}{0}$.

The Problem of the Couriers. Two couriers, A and B, are travelling along the same road in the same direction, *RR'*, at the rates of m and n miles an hour, respectively. If at any time, say 12 o'clock, A is at *P*, and B is a miles beyond him at *Q*, after how many hours, and how many miles beyond *P*, are they together?



Let A and B meet x hours after 12 o'clock, and y miles beyond *P*.

They will then meet $y - a$ miles beyond *Q*.

Since A travels mx miles, and B nx miles, in x hours, we have

$$\left\{ \begin{array}{r} y = mx. \\ y - a = nx. \end{array} \right.$$

Solving these equations, we obtain

$$x = \frac{a}{m-n}, \text{ and } y = \frac{am}{m-n}.$$

We will now discuss these results under different hypotheses.

1. $m > n$.

In this case, the values of x and y are *positive*.

Hence, the couriers will meet at some time *after* 12 o'clock, and at some point to the *right* of P.

This corresponds with the hypothesis made; for if m is greater than n, A is travelling faster than B; and it is evident that he will eventually overtake him at some point beyond their positions at 12 o'clock.

2. $m < n$.

In this case, the values of x and y are *negative*.

Hence, the couriers met at some time *before* 12 o'clock, and at some point to the *left* of P. (Compare § 10.)

This corresponds with the hypothesis made; for if m is less than n, A is travelling more slowly than B; and it is evident that they must have been together before 12 o'clock, and before they could have advanced as far as P.

3. $m = n$, or $m - n = 0$.

In this case, the values of x and y take the forms $\frac{a}{0}$ and $\frac{am}{0}$, respectively.

If $m - n$ approaches the limit 0, x and y increase without limit (§ 294); hence, if $m = n$, no finite values can be assigned to x and y, and the problem is impossible.

Thus, *a result in the form* $\frac{a}{0}$ *indicates that the problem is impossible.*

This interpretation corresponds with the hypothesis made; for if $m = n$, the couriers are a miles apart at 12 o'clock, and are travelling at the same rate; and it is evident that they never could have been, and never will be together.

4. $a = 0$, and $m > n$ or $m < n$.

In this case, $x = 0$ and $y = 0$.

Hence, the couriers are together at 12 o'clock, at P.

This corresponds with the hypothesis made; for if $a = 0$, and m and n are unequal, the couriers are together at 12 o'clock, and are travelling at unequal rates; and it is evident that they never could have been together before 12 o'clock, and never will be together afterwards.

5. $a = 0$, and $m = n$.

In this case, the values of x and y take the form $\frac{0}{0}$.

If $a = 0$, and $m = n$, the couriers are together at 12 o'clock, and are travelling at the same rate.

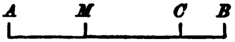
Hence, they always have been, and always will be, together.

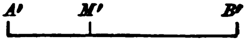
In this case, the number of solutions is indefinitely great; for any value of x whatever, together with the corresponding value of y, will satisfy the given conditions.

Thus, *a result in the form* $\frac{0}{0}$ *indicates that the problem is indeterminate* (§ 293).

THE THEOREM OF LIMITS.

298. *If two variables are always equal, and each approaches a limit, the limits are equal.*





Let AM and $A'M'$ be two variables which are always equal, and approach the limits AB and $A'B'$, respectively.

If possible, suppose $AB > A'B'$, and lay off $AC = A'B'$.

Then the variable AM may assume values between AC and AB, while the variable $A'M'$ is restricted to values less than AC; which is contrary to the hypothesis that the variables should always be equal.

Hence AB cannot be $> A'B'$, and in like manner it may be proved that AB cannot be $< A'B'$; therefore $AB = A'B'$.

XXVII. INDETERMINATE EQUATIONS.

It was shown in § 159 that a single equation which contains two or more unknown quantities is satisfied by an indefinitely great number of sets of values of these quantities. If, however, the unknown quantities are required to satisfy other conditions, the number of solutions may be finite.

We shall consider in the present chapter the solution of indeterminate equations of the first degree, containing two or more unknown quantities, in which the unknown quantities are restricted to *positive integral* values.

299. Solution of Indeterminate Equations in Positive Integers.

1. Solve the equation $7x + 5y = 118$ in positive integers.

Dividing by 5, the smaller of the two coefficients, we have

$$x + \frac{2x}{5} + y = 23 + \frac{3}{5}.$$

Or,
$$\frac{2x-3}{5} = 23 - x - y.$$

Since, by the conditions of the problem, x and y must be positive integers, it follows that $\frac{2x-3}{5}$ must be an integer.

Let this integer be represented by p.

Then,
$$\frac{2x-3}{5} = p, \text{ or } 2x - 3 = 5p. \qquad (1)$$

Dividing (1) by 2,
$$x - 1 - \frac{1}{2} = 2p + \frac{p}{2}.$$

Or,
$$x - 1 - 2p = \frac{p+1}{2}.$$

Since x and p are integers, $x-1-2p$ is also an integer; and therefore $\frac{p+1}{2}$ must be an integer.

Let this integer be represented by q.

Then, $\frac{p+1}{2} = q$, or $p = 2q - 1$.

Substituting in (1), $2x - 3 = 10q - 5$.

Whence, $2x = 10q - 2$, and $x = 5q - 1$. (2)

Substituting this value in the given equation,

$$35q - 7 + 5y = 118.$$

Whence, $5y = 125 - 35q$, and $y = 25 - 7q$. (3)

Equations (2) and (3) form what is called the *general solution in integers* of the given equation.

Now if q is zero, or any negative integer, x will be negative; and if q is any positive integer greater than 3, y will be negative.

Hence, the only *positive integral* values of x and y which satisfy the given equation are those arising from the values 1, 2, 3 of q.

If $q = 1$, $x = 4$, and $y = 18$; if $q = 2$, $x = 9$, and $y = 11$; if $q = 3$, $x = 14$, and $y = 4$.

2. In how many ways can the sum of $15 be paid with dollars, half-dollars, and dimes, the number of dimes being equal to the number of dollars and half-dollars together?

Let x = the number of dollars,
y = the number of half-dollars,
and z = the number of dimes.

Then, $10x + 5y + z = 150$, (1)

and $z = x + y$. (2)

Subtracting (2) from (1),

$$10x + 5y = 150 - x - y, \text{ or } 11x + 6y = 150. \quad (3)$$

Dividing by 6, $x + \frac{5x}{6} + y = 25$.

Then, $\frac{5x}{6}$ must be an integer; or, x must be a multiple of 6.

Let $x = 6p$, where p is an integer.

Substituting in (3), $66p + 6y = 150$, or $y = 25 - 11p$.

Substituting in (2), $z = 6p + 25 - 11p = 25 - 5p$.

The only positive integral solutions are when $p = 1$ or 2; if $p = 1$, $x = 6$, $y = 14$, and $z = 20$; if $p = 2$, $x = 12$, $y = 3$, and $z = 15$.

Then the number of ways is two; either 6 dollars, 14 half-dollars, and 20 dimes; or 12 dollars, 3 half-dollars, and 15 dimes.

EXAMPLES.

Solve the following in positive integers:

3. $2x + 3y = 21.$
4. $7x + 4y = 80.$
5. $7x + 38y = 211.$
6. $31x + 9y = 1222.$
7. $24x + 7y = 422.$
8. $8x + 67y = 158.$
9. $43x + 10y = 719.$
10. $8x + 19y = 700.$
11. $\begin{cases} 2x + 3y - 5z = -8. \\ 5x - y + 4z = 21. \end{cases}$
12. $\begin{cases} 3x - 2y - z = -57. \\ 6x + 11y + 2z = 348. \end{cases}$

Solve the following in least positive integers:

13. $4x - 3y = 5.$
14. $5x - 7y = 11.$
15. $19x - 4y = 128.$
16. $21x - 8y = -25.$
17. $13x - 30y = 61.$
18. $17x - 58y = -79.$

19. In how many different ways can the sum of $ 2.10 be paid with twenty-five and twenty-cent pieces?

20. In how many different ways can the sum of $ 3.90 be paid with fifty and twenty-cent pieces?

21. Find two fractions whose denominators are 9 and 5, respectively, and whose sum shall be equal to $\frac{118}{45}$.

22. In how many different ways can the sum of $ 5.10 be paid with half-dollars, quarter-dollars, and dimes, so that the whole number of coins used shall be 20?

23. A farmer purchased a certain number of pigs, sheep, and calves for $ 160. The pigs cost $ 3 each, the sheep $ 4 each, and the calves $ 7 each; and the number of calves was equal to the number of pigs and sheep together. How many of each did he buy?

24. In how many different ways can the sum of $ 5.45 be paid with quarter-dollars, twenty-cent pieces, and dimes, so that twice the number of quarters plus 5 times the number of twenty-cent pieces shall exceed the number of dimes by 36?

XXVIII. RATIO AND PROPORTION.

300. The **Ratio** of one number to another is the quotient obtained by dividing the first number by the second.

Thus, the ratio of a to b is $\frac{a}{b}$; and it is also expressed $a:b$.

301. A **Proportion** is a statement that two ratios are equal.

The statement that the ratio of a to b is equal to the ratio of c to d, may be written in either of the forms

$$a:b=c:d, \text{ or } \frac{a}{b}=\frac{c}{d}.$$

302. The first and fourth terms of a proportion are called the *extremes*, and the second and third terms the *means*.

The first and third terms are called the *antecedents*, and the second and fourth terms the *consequents*.

Thus, in the proportion $a:b=c:d$, a and d are the extremes, b and c the means, a and c the antecedents, and b and d the consequents.

303. If the means of a proportion are equal, either mean is called a **Mean Proportional** between the first and last terms, and the last term is called a **Third Proportional** to the first and second terms.

Thus, in the proportion $a:b=b:c$, b is a mean proportional between a and c, and c is a third proportional to a and b.

304. A **Fourth Proportional** to three quantities is the fourth term of a proportion whose first three terms are the three quantities taken in their order.

Thus, in the proportion $a : b = c : d$, d is a fourth proportional to a, b, and c.

305. A **Continued Proportion** is a series of equal ratios, in which each consequent is the same as the following antecedent; as,

$$a : b = b : c = c : d = d : e,$$

PROPERTIES OF PROPORTIONS.

306. *In any proportion, the product of the extremes is equal to the product of the means.*

Let the proportion be $a : b = c : d$.

Then by § 301, $\frac{a}{b} = \frac{c}{d}$.

Clearing of fractions, $ad = bc$.

307. *A mean proportional between two quantities is equal to the square root of their product.*

Let the proportion be $a : b = b : c$.

Then, $b^2 = ac$. (§ 306)

Whence, $b = \sqrt{ac}$.

308. From the equation $ad = bc$, we obtain

$$a = \frac{bc}{d}, \text{ and } b = \frac{ad}{c}.$$

That is, in any proportion, either extreme is equal to the product of the means divided by the other extreme; and either mean is equal to the product of the extremes divided by the other mean.

309. (Converse of § 306.) *If the product of two quantities is equal to the product of two others, one pair may be made the extremes, and the other pair the means, of a proportion.*

Let $ad = bc.$

Dividing by bd, $\frac{ad}{bd} = \frac{bc}{bd}$, or $\frac{a}{b} = \frac{c}{d}.$

Whence by § 301, $a : b = c : d.$

In like manner, we may prove that

$$a : c = b : d,$$

$$c : d = a : b, \text{ etc.}$$

310. *In any proportion, the terms are in proportion by* **Alternation**; *that is, the first term is to the third as the second term is to the fourth.*

Let the proportion be $a : b = c : d.$

Then, $ad = bc.$ (§ 306)

Whence, $a : c = b : d.$ (§ 309)

311. *In any proportion, the terms are in proportion by* **Inversion**; *that is, the second term is to the first as the fourth term is to the third.*

Let the proportion be $a : b = c : d.$

Then, $ad = bc.$ (§ 306)

Whence, $b : a = d : c.$ (§ 309)

312. *In any proportion, the terms are in proportion by* **Composition**; *that is, the sum of the first two terms is to the first term as the sum of the last two terms is to the third term.*

Let the proportion be $a : b = c : d.$

Then, $ad = bc.$

Adding each member of the equation to ac,

$$ac + ad = ac + bc.$$

Or, $a(c + d) = c(a + b).$

Whence, $a + b : a = c + d : c.$ (§ 309)

In like manner, we may prove that

$$a + b : b = c + d : d.$$

313. *In any proportion, the terms are in proportion by* **Division**; *that is, the difference of the first two terms is to the first term as the difference of the last two terms is to the third term.*

Let the proportion be $a : b = c : d$.

Then, $ad = bc.$

Subtracting each member of the equation from ac,

$$ac - ad = ac - bc.$$

Or, $a(c - d) = c(a - b).$

Whence, $a - b : a = c - d : c.$

Similarly, $a - b : b = c - d : d.$

314. *In any proportion, the terms are in proportion by* **Composition and Division**; *that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.*

Let the proportion be $a : b = c : d.$

Then by § 312, $$\frac{a+b}{a} = \frac{c+d}{c}. \qquad (1)$$

And by § 313, $$\frac{a-b}{a} = \frac{c-d}{c}. \qquad (2)$$

Dividing (1) by (2), $$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

Whence, $a + b : a - b = c + d : c - d.$

315. *In a series of equal ratios, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let $a : b = c : d = e : f.$

Then by § 306, $ad = bc,$

and $af = be.$

Also, $ab = ba.$

Adding, $a(b + d + f) = b(a + c + e).$

Whence, $a : b = a + c + e : b + d + f.$ (§ 309)

In like manner, the theorem may be proved for any number of equal ratios.

316. *In any proportion, if the first two terms be multiplied by any quantity, as also the last two, the resulting quantities will be in proportion.*

Let the proportion be $a : b = c : d.$

Then, $\frac{a}{b} = \frac{c}{d}.$

Therefore, $\frac{ma}{mb} = \frac{nc}{nd}.$

Whence, $ma : mb = nc : nd.$

In like manner, we may prove that

$$\frac{a}{m} : \frac{b}{m} = \frac{c}{n} : \frac{d}{n}.$$

Note. Either m or n may be unity; that is, either couplet may be multiplied or divided without multiplying or dividing the other.

317. *In any proportion, if the first and third terms be multiplied by any quantity, as also the second and fourth terms, the resulting quantities will be in proportion.*

Let the proportion be $a : b = c : d.$

Then, $\frac{a}{b} = \frac{c}{d}.$

Therefore, $\frac{ma}{nb} = \frac{mc}{nd}.$

Whence, $ma : nb = mc : nd.$

In like manner, we may prove that

$$\frac{a}{m} : \frac{b}{n} = \frac{c}{m} : \frac{d}{n}.$$

Note. Either m or n may be unity.

318. *In any number of proportions, the products of the corresponding terms are in proportion.*

Let the proportions be $a:b=c:d$,

and $e:f=g:h$.

Then, $\frac{a}{b}=\frac{c}{d}$, and $\frac{e}{f}=\frac{g}{h}$.

Multiplying these equals, we have

$$\frac{a}{b}\times\frac{e}{f}=\frac{c}{d}\times\frac{g}{h},\text{ or }\frac{ae}{bf}=\frac{cg}{dh}.$$

Whence, $ae:bf=cg:dh$.

In like manner, the theorem may be proved for any number of proportions.

319. *In any proportion, like powers or like roots of the terms are in proportion.*

Let the proportion be $a:b=c:d$.

Then, $\frac{a}{b}=\frac{c}{d}$.

Therefore, $\frac{a^n}{b^n}=\frac{c^n}{d^n}$.

Whence, $a^n:b^n=c^n:d^n$.

In like manner, we may prove that

$$\sqrt[n]{a}:\sqrt[n]{b}=\sqrt[n]{c}:\sqrt[n]{d}.$$

320. *If three quantities are in continued proportion, the first is to the third as the square of the first is to the square of the second.*

Let $a:b=b:c$.

Then, $\frac{a}{b}=\frac{b}{c}$.

Therefore, $\frac{a}{b}\times\frac{b}{c}=\frac{a}{b}\times\frac{a}{b}$, or $\frac{a}{c}=\frac{a^2}{b^2}$.

Whence, $a:c=a^2:b^2$.

321. *If four quantities are in continued proportion, the first is to the fourth as the cube of the first is to the cube of the second.*

Let $$a : b = b : c = c : d.$$

Then, $$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}.$$

Therefore, $$\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b}, \text{ or } \frac{a}{d} = \frac{a^3}{b^3}.$$

Whence, $$a : d = a^3 : b^3.$$

PROBLEMS.

322. 1. Solve the equation

$$2x + 3 : 2x - 3 = 2b + a : 2b - a.$$

By § 314, $$4x : 6 = 4b : 2a.$$

Dividing the first and third terms by 4, and the second and fourth terms by 2 (§ 317), we have

$$x : 3 = b : a.$$

Whence by § 308, $$x = \frac{3b}{a}, \textit{ Ans.}$$

2. If $x : y = (x + z)^2 : (y + z)^2$, prove that z is a mean proportional between x and y.

From the given proportion, $y(x + z)^2 = x(y + z)^2$. (§ 306)

Or, $$x^2y + 2xyz + yz^2 = xy^2 + 2xyz + xz^2.$$

Or, $$x^2y - xy^2 = xz^2 - yz^2.$$

Dividing by $x - y$, $$xy = z^2.$$

Therefore, z is a mean proportional between x and y (§ 307).

3. If $\frac{a}{b} = \frac{c}{d}$, prove that

$$a^2 - b^2 : a^2 - 3ab = c^2 - d^2 : c^2 - 3cd.$$

Let $\frac{a}{b} = \frac{c}{d} = x$; whence, $a = bx$.

Then, $$\frac{a^2 - b^2}{a^2 - 3ab} = \frac{b^2x^2 - b^2}{b^2x^2 - 3b^2x} = \frac{x^2 - 1}{x^2 - 3x} = \frac{\frac{c^2}{d^2} - 1}{\frac{c^2}{d^2} - \frac{3c}{d}} = \frac{c^2 - d^2}{c^2 - 3cd}.$$

Whence, $$a^2 - b^2 : a^2 - 3ab = c^2 - d^2 : c^2 - 3cd.$$

4. Find a fourth proportional to 35, 20, and 14.

5. Find a mean proportional between 18 and 50.

6. Find a third proportional to $\frac{10}{9}$ and $\frac{5}{12}$.

7. Find the second term of a proportion whose first, third, and fourth terms are $5\frac{1}{7}$, $4\frac{4}{5}$, and $1\frac{5}{9}$.

8. Find a third proportional to $a^2 - 9$ and $a - 3$.

9. Find a mean proportional between $5\frac{5}{6}$ and $18\frac{9}{10}$.

10. Find a mean proportional between

$$\frac{x^2 - x - 6}{x + 4} \text{ and } \frac{x^2 + x - 12}{x + 2}.$$

Solve the following equations:

11. $5x - 3a : 5x + 3a = 7a - 5 : 13a - 5.$

12. $2x - 1 : 3x - 1 = 7x + 1 : 5x - 3.$

13. $x^2 - 16 : x^2 - 25 = x^2 - 2x - 24 : x^2 - 3x - 10.$

14. $1 - \sqrt{1-x} : 1 + \sqrt{1-x} = \sqrt{b} - \sqrt{b-a} : \sqrt{b} + \sqrt{b-a}.$

15. $\begin{cases} ax - by : bx + ay = a^2 - b^2 : 2ab. \\ xy = a^3b^3. \end{cases}$

16. Find two numbers in the ratio 16 to 9 such that, if each be diminished by 8, they shall be in the ratio 12 : 5.

17. Divide 36 into two parts such that the greater diminished by 4 shall be to the less increased by 3 as 3 is to 2.

18. Find two numbers such that, if 4 be added to each, they will be in the ratio 5 to 3; and if 11 be subtracted from each, they will be in the ratio 10 to 3.

19. There are two numbers in the ratio 3 to 4, such that their sum is to the sum of their squares as 7 is to 50. What are the numbers?

20. If $7x - 4z : 8x - 3z = 4y - 7z : 3y - 8z$, prove that z is a mean proportional between x and y.

21. If $ma + nb : pa + qb = mb + nc : pb + qc$, prove that b is a mean proportional between a and c.

22. If $2a - b : 4a + 3b = 2c - d : 4c + 3d$, prove that $a : b = c : d$.

23. If 8 cows and 5 oxen cost four-fifths as much as 9 cows and 7 oxen, what is the ratio of the price of a cow to that of an ox?

24. Given $(a^2 + ab)x + (b^2 - ab)y = (a^2 + b^2)x - (a^2 - b^2)y$; find the ratio of x to y.

25. Find a number such that if it be added to each term of the ratio 5 : 3, the result is $\frac{5}{6}$ of what it would have been if the same number had been subtracted from each term.

If $\frac{a}{b} = \frac{c}{d}$, prove that

26. $2a + 3b : 2a - 3b = 2c + 3d : 2c - 3d$.

27. $a^2 + 2ab : 3ab - 4b^2 = c^2 + 2cd : 3cd - 4d^2$.

28. $a^3 - a^2b + ab^2 : a^3 - b^3 = c^3 - c^2d + cd^2 : c^3 - d^3$.

29. The population of a town increased 2.6 per cent from 1870 to 1880. The number of males decreased 3.8 per cent during the same period, and the number of females increased 10.6 per cent. Find the ratio of males to females in 1870.

30. Each of two vessels contains a mixture of wine and water; in one the wine is to the water as 1 to 3, and in the other the wine is to the water as 3 to 5. A mixture from the two vessels is composed of wine and water in the ratio 9 to 19. Find the ratio of the amounts taken from each vessel.

31. The second of three numbers is a mean proportional between the other two. The third number exceeds the sum of the other two by 15, and the sum of the first and third exceeds twice the second by 12. Find the numbers.

XXIX. VARIATION.

323. One quantity is said to *vary directly* as another when the ratio of any two values of the first is equal to the ratio of the corresponding values of the second.

Note. It is customary to omit the word "directly," and say simply that one quantity *varies* as another.

324. Let us suppose, for example, that a workman receives a fixed sum per day.

The amount which he receives for m days will be to the amount which he receives for n days as m is to n.

That is, the ratio of any two amounts received is equal to the ratio of the corresponding numbers of days worked.

Hence, the amount which the workman receives *varies* as the number of days during which he works.

325. One quantity is said to *vary inversely* as another when the first varies directly as the *reciprocal* of the second.

Thus, the time in which a railway train will traverse a fixed route varies inversely as the speed; that is, if the speed be *doubled*, the train will traverse its route in *one-half* the time.

326. One quantity is said to vary as two others *jointly* when it varies directly as their product.

Thus, the wages of a workman varies jointly as the amount which he receives per day, and the number of days during which he works.

327. One quantity is said to vary directly as a second and inversely as a third, when it varies jointly as the second and the reciprocal of the third.

Thus, in physics, the attraction of a body varies directly as the quantity of matter, and inversely as the square of the distance.

328. The symbol $\propto$ is read "*varies as*"; thus, $a \propto b$ is read "a varies as b."

329. *If $x \propto y$, then x is equal to y multiplied by a constant quantity.*

Let x' and y' denote a *fixed* pair of corresponding values of x and y, and x and y any other pair.

Then by the definition of § 323,

$$\frac{x}{x'}=\frac{y}{y'}, \text{ or } x=\frac{x'}{y'}y.$$

Denoting the constant ratio $\frac{x'}{y'}$ by m, we have

$$x = my.$$

330. It follows from §§ 325, 326, 327, and 329 that:

1. *If x varies inversely as y, $x=\frac{m}{y}$.*
2. *If x varies jointly as y and z, $x = myz$.*
3. *If x varies directly as y and inversely as z, $x=\frac{my}{z}$.*

331. Problems in variation are readily solved by converting the variation into an equation by aid of §§ 329 or 330.

PROBLEMS.

332. 1. If x varies inversely as y, and is equal to 9 when $y = 8$, what is the value of x when $y = 18$?

If x varies inversely as y, we have $x=\frac{m}{y}$ (§ 330).

Putting $x = 9$ and $y = 8$, we obtain $9=\frac{m}{8}$, or $m = 72$.

Then, $x=\frac{72}{y}$; and if $y = 18$, $x=\frac{72}{18}=4$, *Ans.*

2. Given that the area of a triangle varies jointly as its base and altitude, what will be the base of a triangle whose altitude is 12, equivalent to the sum of two triangles whose bases are 10 and 6, and altitudes 3 and 9, respectively?

Let B, H, and A denote the base, altitude, and area, respectively, of any triangle, and B' the base of the required triangle.

Since A varies jointly as B and H, we have $A = mBH$ (§ 330).

Then the area of the first triangle is $m \times 10 \times 3$, or $30\,m$, and the area of the second is $m \times 6 \times 9$, or $54\,m$.

Whence, the area of the required triangle is $30\,m + 54\,m$, or $84\,m$.

But the area of the required triangle is also $m \times B' \times 12$.

Therefore, $12\,mB' = 84\,m$, and $B' = 7$, *Ans.*

3. If $y \propto x$, and is equal to 40 when $x = 5$, what is its value when $x = 9$?

4. If $y \propto z^3$, and is equal to 48 when $z = 4$, what is the expression for y in terms of z^3?

5. If x varies inversely as y, and is equal to $\frac{2}{3}$ when $y = \frac{3}{4}$, what is the value of y when $x = \frac{3}{2}$?

6. If z varies jointly as x and y, and is equal to $\frac{2}{5}$ when $y = \frac{4}{5}$ and $x = \frac{3}{4}$, find the value of z when $x = \frac{4}{3}$ and $y = \frac{5}{4}$.

7. If x varies directly as y and inversely as z, and is equal to $\frac{9}{16}$ when $y = 27$ and $z = 64$, what is the value of x when $y = 9$ and $z = 32$?

8. If $5x + 8 \propto 6y - 1$, and $x = 6$ when $y = -3$, what is the value of x when $y = 7$?

9. If $x^4 \propto y^3$, and $x = 4$ when $y = 4$, what is the value of y when $x = \frac{1}{2}$?

10. The distance fallen by a body from a position of rest varies as the square of the time during which it falls. If it falls $257\frac{1}{3}$ feet in 4 seconds, how far will it fall in 6 seconds?

11. Two quantities vary directly and inversely as x, respectively. If their sum equals $-\frac{11}{12}$ when $x = 1$, and $-\frac{2}{3}$ when $x = -2$, what are the quantities?

12. The area of a circle varies as the square of its diameter. If the area of a circle whose diameter is 4 is $\frac{88}{7}$, what will be the diameter of a circle whose area is $\frac{77}{2}$?

13. If the volume of a pyramid varies jointly as its base and altitude, find the base of a pyramid whose altitude is 11, equivalent to the sum of two pyramids, whose bases are 13 and 14, and altitudes 6 and 7, respectively.

14. Given that y is equal to the sum of two quantities which vary directly as x^2 and inversely as x, respectively. If $y = -\frac{1}{2}$ when $x = 1$, and $y = \frac{37}{4}$ when $x = -2$, what is the value of y when $x = -\frac{1}{2}$?

15. Three spheres of lead whose radii are 6, 8, and 10 inches, respectively, are melted and formed into a single sphere. Find its radius, having given that the volume of a sphere varies as the cube of its radius.

16. The volume of a cone of revolution varies jointly as its altitude and the square of the radius of its base. If the volume of a cone whose altitude is 3 and radius of base 5 is $\frac{550}{7}$, what will be the radius of the base of a cone whose volume is $\frac{880}{7}$ and altitude 5?

17. If 7 men in 4 weeks can earn $238, how many men will earn $127½ in 3 weeks; it being given that the amount earned varies jointly as the number of men, and the number of weeks during which they work?

18. If the volume of a cylinder of revolution varies jointly as its altitude and the square of its radius, what will be the radius of a cylinder whose altitude is 3, equivalent to the sum of two cylinders whose altitudes are 5 and 7, and radii 6 and 3, respectively?

19. If the illumination from a source of light varies inversely as the square of the distance, how much farther from a candle must a book, which is now 15 inches off, be removed, so as to receive just one-third as much light?

20. Given that y is equal to the sum of three quantities, the first of which is constant, and the second and third vary as x and x^3, respectively. If $y = -19$ when $x = 2$, $y = 4$ when $x = 1$, and $y = 2$ when $x = -1$, what is the expression for y in terms of x?

(Represent the constant by l, and the other two quantities by mx and nx^3.)

XXX. PROGRESSIONS.

ARITHMETIC PROGRESSION.

333. An **Arithmetic Progression** is a series of terms each of which is derived from the preceding by adding a constant quantity called the *common difference.*

Thus, 1, 3, 5, 7, 9, 11, ⋯ is an arithmetic progression in which the common difference is 2.

Again, 12, 9, 6, 3, 0, -3, ⋯ is an arithmetic progression in which the common difference is -3.

334. *Given the first term, a, the common difference, d, and the number of terms, n, to find the last term, l.*

The progression is $a,\ a+d,\ a+2d,\ a+3d,\ \cdots$.

It will be observed that the coefficient of d in any term is 1 less than the number of the term.

Then in the nth or last term the coefficient of d is $n-1$.

That is, $$l = a + (n-1)d. \qquad \text{(I.)}$$

335. *Given the first term, a, the last term, l, and the number of terms, n, to find the sum of the terms, S.*

$$S = a + (a+d) + (a+2d) + \cdots + (l-d) + l.$$

Writing the terms in reverse order,

$$S = l + (l-d) + (l-2d) + \cdots + (a+d) + a.$$

Adding these equations term by term,

$$2S = (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l).$$

Therefore, $2S = n(a+l)$, and $S = \frac{n}{2}(a+l)$. (II.)

336. Substituting in (II.) the value of l from (I.), we have $$S = \frac{n}{2}[2a + (n-1)d].$$

EXAMPLES.

337. 1. Find the last term and the sum of the terms of the progression 8, 5, 2, ⋯ to 27 terms.

In this case, $a = 8$, $d = 5 - 8 = -3$, and $n = 27$.

Substituting in (I.), $l = 8 + (27 - 1)(-3) = 8 - 78 = -70$.

Substituting in (II.), $S = \frac{27}{2}(8 - 70) = 27 \times (-31) = -837$.

Note. The common difference may be found by subtracting the first term from the second, or any term from the next following term.

Find the last term and the sum of the terms of:

2. 3, 9, 15, ⋯ to 12 terms.
3. -7, -12, -17, ⋯ to 15 terms.
4. -69, -62, -55, ⋯ to 16 terms.
5. $\frac{3}{4}$, $-\frac{3}{8}$, $-\frac{3}{2}$, ⋯ to 17 terms.
6. $\frac{5}{4}$, $\frac{43}{12}$, $\frac{71}{12}$, ⋯ to 13 terms.
7. $-\frac{1}{3}$, $\frac{1}{2}$, $\frac{4}{3}$, ⋯ to 22 terms.
8. $-\frac{3}{4}$, $-\frac{5}{6}$, $-\frac{11}{12}$, ⋯ to 55 terms.
9. $-\frac{6}{5}$, $-\frac{3}{2}$, $-\frac{9}{5}$, ⋯ to 19 terms.
10. $2a - 5b$, $6a - 2b$, $10a + b$, ⋯ to 9 terms.
11. $\frac{x-y}{2}$, $\frac{y}{2}$, $\frac{3y-x}{2}$, ⋯ to 10 terms.

338. If any three of the five elements of an arithmetic progression are given, the other two may be found by substituting the given values in the fundamental formulæ (I.) and (II.), and solving the resulting equations.

1. Given $a = -\frac{5}{3}$, $n = 20$, $S = -\frac{5}{3}$; find d and l.

Substituting the given values in (II.), we have

$$-\frac{5}{3} = 10\left(-\frac{5}{3} + l\right), \text{ or } -\frac{1}{6} = -\frac{5}{3} + l; \text{ whence, } l = \frac{5}{3} - \frac{1}{6} = \frac{3}{2}.$$

Substituting the values of l, a, and n in (I.), we have

$$\frac{3}{2} = -\frac{5}{3} + 19d; \text{ whence, } 19d = \frac{3}{2} + \frac{5}{3} = \frac{19}{6}, \text{ and } d = \frac{1}{6}.$$

2. Given $d = -3$, $l = -39$, $S = -264$; find a and n.

Substituting in (I.), $-39 = a + (n-1)(-3)$, or $a = 3n - 42$. (1)

Substituting the values of S, a, and l in (II.), we have

$$-264 = \frac{n}{2}(3n - 42 - 39), \text{ or } -528 = 3n^2 - 81n, \text{ or } n^2 - 27n = -176.$$

Whence, $$n = \frac{27 \pm \sqrt{729 - 704}}{2} = \frac{27 \pm 5}{2} = 16 \text{ or } 11.$$

Substituting in (1), $a = 48 - 42$ or $33 - 42 = 6$ or -9.

Therefore, $a = 6$ and $n = 16$; or, $a = -9$ and $n = 11$, *Ans.*

Note 1. The interpretation of the two answers is as follows:

If $a = 6$ and $n = 16$, the progression is

6, 3, 0, −3, −6, −9, −12, −15, −18, −21, −24, −27, −30, −33, −36, −39.

If $a = -9$ and $n = 11$, the progression is

−9, −12, −15, −18, −21, −24, −27, −30, −33, −36, −39.

In each of these the sum is −264.

3. Given $a = \frac{1}{3}$, $d = -\frac{1}{12}$, $S = -\frac{3}{2}$; find l and n.

Substituting in (I.), $$l = \frac{1}{3} + (n-1)\left(-\frac{1}{12}\right) = \frac{5-n}{12}. \qquad (1)$$

Substituting the values of S, a, and l in (II.), we have

$$-\frac{3}{2} = \frac{n}{2}\left(\frac{1}{3} + \frac{5-n}{12}\right), \text{ or } -3 = n\left(\frac{9-n}{12}\right), \text{ or } n^2 - 9n = 36.$$

Whence, $$n = \frac{9 \pm \sqrt{81 + 144}}{2} = \frac{9 \pm 15}{2} = 12 \text{ or } -3.$$

The value $n = -3$ is inapplicable, for the number of terms in a progression must be a *positive integer*.

Substituting the value $n = 12$ in (1), $l = \frac{5 - 12}{12} = -\frac{7}{12}$.

Therefore, $l = -\frac{7}{12}$ and $n = 12$, *Ans.*

Note 2. A *negative* or *fractional* value of n is inapplicable, and must be rejected, together with all other values dependent upon it.

EXAMPLES.

4. Given $d = 5,\ l = 71,\ n = 15$; find a and S.
5. Given $d = -4,\ n = 20,\ S = -620$; find a and l.
6. Given $a = -9,\ n = 23,\ l = 57$; find d and S.
7. Given $a = -5,\ n = 19,\ S = -950$; find d and l.
8. Given $a = \frac{1}{4},\ l = \frac{35}{4},\ S = \frac{315}{2}$; find d and n.
9. Given $l = -\frac{3}{5},\ n = 19,\ S = 0$; find a and d.
10. Given $d = \frac{1}{12},\ S = \frac{62}{3},\ a = \frac{2}{3}$; find l and n.
11. Given $a = \frac{1}{2},\ l = -\frac{5}{11},\ d = -\frac{1}{22}$; find n and S.
12. Given $d = \frac{1}{2},\ n = 17,\ S = 17$; find a and l.
13. Given $l = 6,\ d = \frac{5}{6},\ S = 24$; find a and n.
14. Given $l = -5\frac{1}{6},\ n = 21,\ S = -38\frac{1}{2}$; find a and d.
15. Given $a = -\frac{5}{2},\ l = -\frac{23}{2},\ S = -91$; find d and n.
16. Given $a = \frac{3}{4},\ n = 15,\ S = \frac{405}{8}$; find d and l.
17. Given $a = \frac{15}{2},\ d = -\frac{3}{4},\ S = \frac{135}{4}$; find n and l.

18. Given $l = -\frac{4}{3}$, $d = -\frac{1}{15}$, $S = -\frac{40}{3}$; find a and n.

19. Given $a = 5$, $d = -\frac{4}{3}$, $S = -80$; find n and l.

From (I.) and (II.), *general formulæ* for the solution of examples like the above may be readily derived.

20. Given a, d, and S; derive the formula for n.

By § 336, $2S = n[2a + (n-1)d]$, or $dn^2 + (2a - d)n = 2S$.

This is a quadratic in n; and may be solved by the method of § 261

Multiplying by $4d$, and adding $(2a - d)^2$ to both members,

$$4d^2n^2 + 4d(2a - d)n + (2a - d)^2 = 8dS + (2a - d)^2.$$

Extracting the square root,

$$2dn + 2a - d = \pm\sqrt{8dS + (2a - d)^2}.$$

Whence, $$n = \frac{d - 2a \pm \sqrt{8dS + (2a - d)^2}}{2d}, \textit{ Ans.}$$

21. Given a, l, and n; derive the formula for d.

22. Given a, n, and S; derive the formulæ for d and l.

23. Given d, n, and S; derive the formulæ for a and l.

24. Given a, d, and l; derive the formulæ for n and S.

25. Given d, l, and n; derive the formulæ for a and S.

26. Given l, n, and S; derive the formulæ for a and d.

27. Given a, d, and S; derive the formula for l.

28. Given a, l, and S; derive the formulæ for d and n.

29. Given d, l, and S; derive the formulæ for a and n.

339. *To insert any number of arithmetic means between two given terms.*

1. Insert 5 arithmetic means between 3 and -5.

We are to find an arithmetic progression of 7 terms, whose first term is 3, and last term -5.

Putting $a = 3$, $l = -5$, and $n = 7$, in (I.), § 334, we have

$$-5 = 3 + 6d;\ \text{whence, } 6d = -8, \text{ and } d = -\frac{4}{3}.$$

Hence, the required progression is

$$3,\ \frac{5}{3},\ \frac{1}{3},\ -1,\ -\frac{7}{3},\ -\frac{11}{3},\ -5,\ \textit{Ans.}$$

EXAMPLES.

2. Insert 6 arithmetic means between 3 and 8.

3. Insert 4 arithmetic means between $\frac{10}{3}$ and $-\frac{5}{2}$.

4. Insert 5 arithmetic means between $-\frac{4}{3}$ and 1.

5. Insert 7 arithmetic means between $-\frac{3}{2}$ and $\frac{9}{2}$.

6. Insert 8 arithmetic means between $-\frac{5}{4}$ and -5.

7. Insert 9 arithmetic means between $\frac{3}{2}$ and -11.

340. Let x denote the arithmetic mean between a and b Then, by the nature of the progression,

$$x - a = b - x, \text{ or } 2x = a + b.$$

Whence, $$x = \frac{a+b}{2}.$$

That is, *the arithmetic mean between two quantities is equa to one-half their sum.*

EXAMPLES.

Find the arithmetic mean between:

1. $\frac{5}{12}$ and $-\frac{3}{20}$.

2. $(x+7)^2$ and $(x-7)^2$.

3. $\frac{2a-1}{2a+1}$ and $\frac{2a+1}{2a-1}$.

4. $\frac{a+b}{a-b}$ and $-\frac{a^3+b^3}{a^3-b^2}$.

PROBLEMS.

341. 1. The sixth term of an arithmetic progression is $\frac{5}{6}$, and the fifteenth term is $\frac{16}{3}$. Find the first term.

By § 334, the sixth term is $a + 5d$, and the fifteenth term $a + 14d$.

Then by the conditions,
$$\begin{cases} a + 5d = \frac{5}{6}. & (1) \\ a + 14d = \frac{16}{3}. & (2) \end{cases}$$

Subtracting (1) from (2), $9d = \frac{9}{2}$; whence, $d = \frac{1}{2}$.

Substituting in (1), $a + \frac{5}{2} = \frac{5}{6}$; whence, $a = -\frac{5}{3}$, *Ans.*

2. Find four numbers in arithmetic progression such that the product of the first and fourth shall be 45, and the product of the second and third 77.

Let the numbers be $x - 3y$, $x - y$, $x + y$, and $x + 3y$.

Then by the conditions, $\begin{cases} x^2 - 9y^2 = 45. \\ x^2 - y^2 = 77. \end{cases}$

Solving these equations, $x = 9$, $y = \pm 2$; or, $x = -9$, $y = \pm 2$ (§ 276).

Then the numbers are 3, 7, 11, 15; or, -3, -7, -11, -15.

Note. In problems like the above, it is convenient to represent the unknown quantities by *symmetrical* expressions.

Thus, if five numbers had been required to be found, we should have represented them by $x - 2y$, $x - y$, x, $x + y$, and $x + 2y$.

3. Find the sum of all the integers beginning with 1 and ending with 100.

4. Find the sum of all the even integers beginning with 2 and ending with 1000.

5. The 8th term of an arithmetic progression is 10, and the 14th term is -14. Find the 23d term.

6. Find four numbers in arithmetic progression such that the sum of the first two shall be 12, and the sum of the last two -20.

7. Find the sum of the first 15 positive integers which are multiples of 7.

8. The 19th term of an arithmetic progression is $9x-2y$, and the 31st term is $13x-8y$. Find the sum of the first thirteen terms.

9. Find four integers in arithmetic progression such that their sum shall be 24, and their product 945.

10. How many positive integers of three digits are there which are multiples of 9?

11. Find the sum of all positive integers of three digits which are multiples of 11.

12. The 7th term of an arithmetic progression is $-\frac{1}{6}$, the 16th term is $\frac{7}{3}$, and the last term is $\frac{13}{2}$. Find the number of terms.

13. The sum of the 2d and 6th terms of an arithmetic progression is $-\frac{5}{2}$, and the sum of the 5th and 9th terms is -10. Find the first term.

14. Find five numbers in arithmetic progression such that the sum of the second, third, and fifth shall be 10, and the product of the first and fourth -36.

15. If m arithmetic means be inserted between a and b, what is the first mean?

16. How many positive integers of one, two, or three digits are there which are multiples of 8?

17. How many arithmetic means are inserted between 4 and 36, when the second mean is to the first as 4 is to 3?

18. A man travels 3 miles the first day, 6 miles the second day, 9 miles the third day, and so on. After he has travelled a certain number of days, he finds his average daily distance to be 46½ miles. How many days has he been travelling?

19. How many arithmetic means are inserted between $\frac{3}{5}$ and $-\frac{9}{7}$, when the sum of the first two is $\frac{9}{35}$?

20. After A had travelled for $4\frac{1}{2}$ hours at the rate of 5 miles an hour, B set out to overtake him, and travelled 3 miles the first hour, $3\frac{1}{2}$ miles the second hour, 4 miles the third hour, and so on; in how many hours will B overtake A?

21. Find three numbers in arithmetic progression such that the sum of their squares is 347, and one-half the third number exceeds the sum of the first and second by $4\frac{1}{2}$.

22. The digits of a number of three figures are in arithmetic progression; the sum of the first two digits exceeds the third by 3; and if 396 be added to the number, the digits will be inverted. Find the number.

GEOMETRIC PROGRESSION.

342. A **Geometric Progression** is a series of terms each of which is derived from the preceding by multiplying by a constant quantity called the *ratio.*

Thus, 2, 6, 18, 54, 162, $\cdots$ is a geometric progression in which the ratio is 3.

Again, 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, $\cdots$ is a geometric progression in which the ratio is $\frac{1}{3}$.

Negative values of the ratio are also admissible.

Thus, $-3, 6, -12, 24, -48, \cdots$ is a geometric progression in which the ratio is -2.

343. *Given the first term, a, the ratio, r, and the number of terms, n, to find the last term, l.*

The progression is $a, ar, ar^2, ar^3, \cdots$.

It will be observed that the exponent of r in any term is 1 less than the number of the term.

Then in the nth or last term the exponent of r is $n-1$.

That is, $$l = ar^{n-1}. \qquad \text{(I.)}$$

344. *Given the first term, a, the last term, l, and the ratio, r, to find the sum of the terms, S.*

$$S = a + ar + ar^2 + \cdots + ar^{n-3} + ar^{n-2} + ar^{n-1}.$$

Multiplying each term by r, we have

$$rS = ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} + ar^n.$$

Subtracting the first equation from the second,

$$rS - S = ar^n - a.$$

Whence, $$S = \frac{ar^n - a}{r - 1}.$$

But by (I.), § 343, $rl = ar^n$.

Therefore, $$S = \frac{rl - a}{r - 1}. \qquad \text{(II.)}$$

EXAMPLES.

345. 1. Find the last term and the sum of the terms of the progression $3,\ 1,\ \frac{1}{3}, \cdots$ to 7 terms.

In this case, $a = 3$, $r = \frac{1}{3}$, and $n = 7$.

Substituting in (I.), $l = 3\left(\frac{1}{3}\right)^6 = \frac{1}{3^5} = \frac{1}{243}$.

Substituting in (II.), $S = \dfrac{\frac{1}{3} \times \frac{1}{243} - 3}{\frac{1}{3} - 1} = \dfrac{\frac{1}{729} - 3}{-\frac{2}{3}} = \dfrac{-\frac{2186}{729}}{-\frac{2}{3}} = \dfrac{1093}{243}$.

Note. The ratio may be found by dividing the second term by the first, or any term by the next preceding term.

2. Find the last term and the sum of the terms of the progression $-2,\ 6,\ -18, \cdots$ to 8 terms.

In this case, $a = -2$, $r = \frac{6}{-2} = -3$, and $n = 8$.

Then, $l = -2(-3)^7 = -2 \times (-2187) = 4374$.

And, $S = \dfrac{-3 \times 4374 - (-2)}{-3 - 1} = \dfrac{-13122 + 2}{-4} = 3280$.

Find the last term and the sum of the terms of:

3. $1, 3, 9, \cdots$ to 8 terms.

4. $6, 4, \frac{8}{3}, \cdots$ to 7 terms.

5. $-2, 10, -50, \cdots$ to 5 terms.

6. $2, 4, 8, \cdots$ to 11 terms.

7. $-3, \frac{3}{2}, -\frac{3}{4}, \cdots$ to 9 terms.

8. $-\frac{5}{2}, -5, -10, \cdots$ to 10 terms.

9. $-5, 2, -\frac{4}{5}, \cdots$ to 6 terms.

10. $-\frac{1}{3}, \frac{1}{2}, -\frac{3}{4}, \cdots$ to 7 terms.

11. $\frac{2}{3}, \frac{1}{2}, \frac{3}{8}, \cdots$ to 5 terms.

12. $-\frac{3}{4}, 3, -12, \cdots$ to 6 terms.

346. If any three of the five elements of a geometric progression are given, the other two may be found by substituting the given values in the fundamental formulæ (I.) and (II.), and solving the resulting equations.

But in certain cases the operation involves the solution of an equation of a degree higher than the second; and in others the unknown quantity appears as an exponent, the solution of which form of equation can usually only be affected by the aid of logarithms (§ 419).

In all such cases in the present chapter, the equations may be solved by inspection.

1. Given $a = -2$, $n = 5$, $l = -32$; find r and S.

Substituting the given values in (I.), we have

$$-32 = -2r^4\text{; whence, } r^4 = 16, \text{ and } r = \pm 2.$$

Substituting in (II.),

If $r = 2$, $S = \frac{2(-32)-(-2)}{2-1} = -64 + 2 = -62$.

If $r = -2$, $S = \frac{(-2)(-32)-(-2)}{-2-1} = \frac{64+2}{-3} = -22$.

Therefore, $r = 2$ and $S = -62$; or, $r = -2$ and $S = -22$, *Ans.*

Note 1. The interpretation of the two answers is as follows:

If $r = 2$, the progression is $-2, -4, -8, -16, -32$, whose sum is -62.

If $r = -2$, the progression is $-2, 4, -8, 16, -32$, whose sum is -22.

2. Given $a = 3$, $r = -\frac{1}{3}$, $S = \frac{1640}{729}$; find n and l.

Substituting in (II.), $\frac{1640}{729} = \frac{-\frac{1}{3}l - 3}{-\frac{1}{3} - 1} = \frac{l+9}{4}$.

Whence, $l + 9 = \frac{6560}{729}$; or, $l = \frac{6560}{729} - 9 = -\frac{1}{729}$.

Substituting the values of l, a, and r in (I.), we have

$$-\frac{1}{729} = 3\left(-\frac{1}{3}\right)^{n-1}; \text{ or, } \left(-\frac{1}{3}\right)^{n-1} = -\frac{1}{2187}.$$

Whence, by inspection, $n - 1 = 7$, or $n = 8$.

EXAMPLES.

3. Given $r = 2$, $n = 9$, $l = 256$; find a and S.

4. Given $r = \frac{2}{3}$, $n = 5$, $S = \frac{211}{27}$; find a and l.

5. Given $a = -2$, $n = 6$, $l = 2048$; find r and S.

6. Given $a = 2$, $r = -\frac{1}{2}$, $l = -\frac{1}{256}$; find n and S.

7. Given $r = \frac{1}{2}$, $n = 11$, $S = \frac{2047}{2048}$; find a and l.

8. Given $a = \frac{2}{3}$, $n = 9$, $l = \frac{2187}{128}$; find r and S.

9. Given $a = -8$, $l = -\frac{1}{32}$, $S = -\frac{511}{32}$; find r and n.

10. Given $a = \frac{3}{4}$, $r = -\frac{1}{3}$, $S = \frac{91}{162}$; find l and n.

11. Given $l = 192$, $r = -2$, $S = 129$; find a and n.

12. Given $a = -\frac{2}{3}$, $l = -\frac{1}{192}$, $S = -\frac{255}{192}$; find r and n.

From (I.) and (II.), general formulæ may be derived for the solution of cases like the above.

13. Given a, r, and S; derive the formula for l.

14. Given a, l, and S; derive the formula for r.

15. Given r, l, and S; derive the formula for a.

16. Given r, n, and l; derive the formulæ for a and S.

17. Given r, n, and S; derive the formulæ for a and l.

18. Given a, n, and l; derive the formulæ for r and S.

Note 2. If the given elements are n, l, and S, *equations* for a and r may be found, but there are no definite *formulæ* for their values. The same is the case when the given elements are a, n, and S.

The general formulæ for n involve logarithms; these cases are discussed in § 419.

347. The limit (§ 292) to which the sum of the terms of a *decreasing* geometric progression approaches, when the number of terms is indefinitely increased, is called the *sum of the series to infinity.*

Formula (II.), § 344, may be written

$$S = \frac{a - rl}{1 - r}.$$

It is evident that, by sufficiently continuing a decreasing geometric progression, the last term may be made numerically less than any assigned number, however small.

Hence, when the number of terms is indefinitely increased, l, and therefore rl, approaches the limit 0.

Then the fraction $\frac{a-rl}{1-r}$ approaches the limit $\frac{a}{1-r}$.

Therefore, the sum of a decreasing geometric progression to infinity is given by the formula

$$S=\frac{a}{1-r}. \qquad \text{(III.)}$$

EXAMPLES.

1. Find the sum of the series $4, -\frac{8}{3}, \frac{16}{9}, \cdots$, to infinity.

In this case, $a=4$, $r=-\frac{2}{3}$.

Substituting in (III.), $S=\frac{4}{1+\frac{2}{3}}=\frac{12}{5}$, *Ans.*

Find the sum of the following to infinity:

2. $3, 1, \frac{1}{3}, \cdots$.

3. $16, -4, 1, \cdots$.

4. $-1, \frac{1}{5}, -\frac{1}{25}, \cdots$.

5. $-\frac{5}{3}, -\frac{10}{9}, -\frac{20}{27}, \cdots$.

6. $\frac{7}{4}, \frac{21}{32}, \frac{63}{256}, \cdots$.

7. $\frac{2}{5}, -\frac{1}{3}, \frac{5}{18}, \cdots$.

8. $-\frac{1}{8}, -\frac{1}{18}, -\frac{2}{81}, \cdots$.

9. $\frac{5}{7}, -\frac{5}{8}, \frac{35}{64}, \cdots$.

348. *To find the value of a repeating decimal.*

This is a case of finding the sum of a decreasing geometric series to infinity, and may be solved by formula (III.).

1. Find the value of $.85151\cdots$.

We have, $.85151\cdots=.8+.051+.00051+\cdots$.

The terms after the first constitute a decreasing geometric progression, in which $a=.051$ and $r=.01$.

Substituting in (III.), $S=\frac{.051}{1-.01}=\frac{.051}{.99}=\frac{51}{990}=\frac{17}{330}$.

Then the value of the given decimal is $\frac{8}{10}+\frac{17}{330}$, or $\frac{281}{330}$, *Ans.*

EXAMPLES.

Find the values of the following:

2. .8181 ….	**4.** .69444 ….	**6.** .11567567 ….
3. .296296 ….	**5.** .58686 ….	**7.** .922828 ….

349. *To insert any number of geometric means between two given terms.*

1. Insert 5 geometric means between 2 and $\frac{128}{729}$.

We are to find a geometric progression of 7 terms, whose first term is 2, and last term $\frac{128}{729}$.

Putting $a = 2$, $l = \frac{128}{729}$, and $n = 7$, in (I.), § 343, we have

$$\frac{128}{729} = 2r^6;\ \text{whence, } r^6 = \frac{64}{729},\ \text{and } r = \pm\frac{2}{3}.$$

Hence, the required result is

$$2,\ \pm\frac{4}{3},\ \frac{8}{9},\ \pm\frac{16}{27},\ \frac{32}{81},\ \pm\frac{64}{243},\ \frac{128}{729},\ \textit{Ans.}$$

EXAMPLES.

2. Insert 4 geometric means between 3 and 729.

3. Insert 6 geometric means between $\frac{1}{6}$ and $-\frac{64}{3}$.

4. Insert 5 geometric means between 2 and 128.

5. Insert 3 geometric means between $-\frac{2}{5}$ and $-\frac{125}{8}$.

6. Insert 4 geometric means between $-\frac{7}{2}$ and 3584.

7. Insert 7 geometric means between $\frac{243}{128}$ and $\frac{2}{27}$.

350. Let x denote the geometric mean between a and b.

Then, by the nature of the progression, $\frac{x}{a} = \frac{b}{x}$, or $x^2 = ab$.

Whence, $x = \sqrt{ab}$.

That is, *the geometric mean between two quantities is equal to the square root of their product.*

EXAMPLES.

Find the geometric mean between:

1. $2\frac{11}{12}$ and $1\frac{13}{15}$. 2. $9 + 4\sqrt{5}$ and $9 - 4\sqrt{5}$.

3. $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$.

4. $\frac{2x^2 + 4xy}{xy - 2y^2}$ and $\frac{xy + 2y^2}{2x^2 - 4xy}$.

PROBLEMS.

351. 1. Find three numbers in geometric progression such that their sum shall be 14, and the sum of their squares 84.

Let the numbers be a, ar, and ar^2.

Then by the conditions, $\begin{cases} a + ar + ar^2 = 14. & (1) \\ a^2 + a^2r^2 + a^2r^4 = 84. & (2) \end{cases}$

Dividing (2) by (1), $a - ar + ar^2 = 6.$ (3)

Subtracting (3) from (1), $2ar = 8$, or $r = \frac{4}{a}$. (4)

Substituting in (1), $a + 4 + \frac{16}{a} = 14$, or $a^2 - 10a = -16$.

Solving this equation, $a = 8$ or 2.

Substituting in (4), $r = \frac{4}{8}$ or $\frac{4}{2} = \frac{1}{2}$ or 2.

Therefore, the numbers are 2, 4, and 8, *Ans.*

2. The 4th term of a geometric progression is $-\frac{16}{9}$, and the 7th term is $\frac{128}{243}$. Find the second term.

3. The sum of the first and last of four numbers in geometric progression is 112, and the sum of the second and third is 48. Find the numbers.

4. The product of three numbers in geometric progression is -1000, and the sum of the squares of the second and third is 500. Find the numbers.

5. A man saves every year half as much again as he saved the preceding year. If he saved \$ 128 the first year, to what sum will his savings amount at the end of seven years?

6. A body moves 12 feet the first second, and in each succeeding second five-eighths as far as in the preceding second, until it comes to rest. How far will it have moved?

7. The 5th term of a geometric progression is $-\frac{9}{4}$, and the 9th term is $-\frac{729}{64}$. Find the 11th term.

8. If m geometric means be inserted between a and b, what is the first mean?

9. The sum of three numbers in arithmetic progression is 12. If the first number be increased by 5, the second by 2, and the third by 7, the resulting numbers form a geometric progression. What are the numbers?

10. Divide \$ 700 between A, B, C, and D, so that their shares may be in geometric progression, and the sum of A's and B's shares equal to \$ 252.

11. There are four numbers, the first three of which form an arithmetic progression, and the last three a geometric progression. The sum of the first and third is 2, and of the second and fourth 37. What are the numbers?

12. Find the ratio of the geometric progression in which the sum of the first ten terms is 244 times the sum of the first five terms.

13. There are three numbers in geometric progression whose sum is 19. If the first be multiplied by $\frac{5}{2}$, the second by $\frac{4}{3}$, and the third by $\frac{2}{3}$, the resulting numbers form an arithmetic progression. What are the numbers?

HARMONIC PROGRESSION.

352. A **Harmonic Progression** is a series of terms whose reciprocals form an arithmetic progression.

Thus, $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \cdots$ is a harmonic progression, because the reciprocals of the terms, $1, 3, 5, 7, 9, \cdots$, form an arithmetic progression.

353. Any problem in harmonic progression which is susceptible of solution, may be solved by taking the reciprocals of the terms, and applying the formulæ of the arithmetic progression. There is, however, no general method for finding the *sum of the terms* of a harmonic progression.

354. Let x denote the harmonic mean between a and b.

Then, $\frac{1}{x}$ is the *arithmetic* mean between $\frac{1}{a}$ and $\frac{1}{b}$ (§ 352).

Whence by § 340, $\frac{1}{x} = \frac{\frac{1}{a}+\frac{1}{b}}{2} = \frac{a+b}{2ab}$, and $x = \frac{2ab}{a+b}$.

EXAMPLES.

355. 1. Find the last term of the progression $2, \frac{2}{3}, \frac{2}{5}, \cdots$ to 36 terms.

Taking the reciprocals of the terms, we have the arithmetic progression $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$.

In this case, $a = \frac{1}{2}$, $d = 1$, and $n = 36$.

Substituting in (I.), § 334, we have $l = \frac{1}{2} + (36 - 1) \times 1 = \frac{71}{2}$.

Taking the reciprocal of this, the last term of the given harmonic progression is $\frac{2}{71}$, *Ans.*

2. Insert 5 harmonic means between 2 and -3.

We have to insert 5 arithmetic means between $\frac{1}{2}$ and $-\frac{1}{3}$.

Putting $a = \frac{1}{2}$, $l = -\frac{1}{3}$, and $n = 7$, in (I.), § 334, we have

$$-\frac{1}{3} = \frac{1}{2} + 6d\text{; whence, } 6d = -\frac{5}{6}, \text{ or } d = -\frac{5}{36}.$$

Then the arithmetic progression is

$$\frac{1}{2}, \frac{13}{36}, \frac{2}{9}, \frac{1}{12}, -\frac{1}{18}, -\frac{7}{36}, -\frac{1}{3}.$$

Therefore, the required harmonic progression is

$$2, \frac{36}{13}, \frac{9}{2}, 12, -18, -\frac{36}{7}, -3, \textit{ Ans.}$$

Find the last terms of the following:

3. $\frac{2}{5}, \frac{4}{7}, 1, \cdots$ to 13 terms. **4.** $\frac{4}{5}, \frac{12}{43}, \frac{12}{71}, \cdots$ to 25 terms.

5. $-3, 2, \frac{3}{4}, \cdots$ to 38 terms.

6. $-\frac{4}{3}, -\frac{6}{5}, -\frac{12}{11}, \cdots$ to 43 terms.

7. $-\frac{5}{6}, -\frac{2}{3}, -\frac{5}{9}, \cdots$ to 17 terms.

8. Insert 6 harmonic means between 2 and $-\frac{10}{9}$.

9. Insert 7 harmonic means between $-\frac{2}{5}$ and $\frac{2}{7}$.

10. Insert 8 harmonic means between $-\frac{4}{5}$ and $-\frac{1}{5}$.

Find the harmonic mean between:

11. 3 and 6. **12.** $\frac{1-x}{1+x}$ and $-\frac{1-x^2}{1+x^2}$.

13. The first term of a harmonic progression is x, and the second term is y; continue the series to three more terms.

14. The arithmetic mean between two numbers is 1, and the harmonic mean -15. Find the numbers.

15. The 5th term of a harmonic progression is $-\frac{4}{5}$, and the 11th term is $-\frac{1}{8}$. What is the 15th term?

16. Prove that, if a, b, and c are in harmonic progression,

$$a : c = a - b : b - c.$$

XXXI. THE BINOMIAL THEOREM.

POSITIVE INTEGRAL EXPONENT.

356. The **Binomial Theorem** is a formula by means of which any power of a binomial, positive or negative, integral or fractional, may be expanded into a series.

We shall consider in the present chapter those cases only in which the exponent is a *positive integer*.

357. Proof of the Binomial Theorem for a Positive Integral Exponent.

By actual multiplication, we obtain:

$$(a+x)^2 = a^2 + 2ax + x^2;$$

$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3;$$

$$(a+x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4; \text{ etc.}$$

In the above results, we observe the following laws:

1. The number of terms is greater by 1 than the exponent of the binomial.

2. The exponent of a in the first term is the same as the exponent of the binomial, and decreases by 1 in each succeeding term.

3. The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

4. The coefficient of the first term is 1, and the coefficient of the second term is the exponent of the binomial.

5. If the coefficient of any term be multiplied by the exponent of a in that term, and the result divided by the ~~exponent of x in~~ the term i~~ncreased by 1~~, the quotient will be the coefficient of the next following term.

6. [illegible]

358. If the laws of § 357 be assumed to hold for the expansion of $(a+x)^n$, where n is any positive integer, the exponent of a in the first term is n, in the second term $n-1$, in the third term $n-2$, in the fourth term $n-3$, etc.

The exponent of x in the second term is 1, in the third term 2, in the fourth term 3, etc.

The coefficient of the first term is 1; of the second term n.

Multiplying the coefficient of the second term, n, by $n-1$, the exponent of a in that term, and dividing the result by the exponent of x in the term increased by 1, or 2, we have $\frac{n(n-1)}{1\cdot 2}$* as the coefficient of the third term; and so on.

Then,
$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3 + \cdots. \qquad (1)$$

Multiplying both members of (1) by $a+x$, we have

$$\begin{aligned}(a+x)^{n+1} = a^{n+1} &+ na^nx + \frac{n(n-1)}{1\cdot 2}a^{n-1}x^2 \\ &+ \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-2}x^3 + \cdots \\ &+ a^nx + na^{n-1}x^2 + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^3 + \cdots.\end{aligned}$$

Collecting the terms which contain like powers of a and x,

$$\begin{aligned}(a+x)^{n+1} &= a^{n+1} + (n+1)a^nx + \left[\frac{n(n-1)}{1\cdot 2} + n\right]a^{n-1}x^2 \\ &\quad + \left[\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} + \frac{n(n-1)}{1\cdot 2}\right]a^{n-2}x^3 + \cdots \\ &= a^{n+1} + (n+1)a^nx + n\left[\frac{n-1}{2} + 1\right]a^{n-1}x^2 \\ &\quad + \frac{n(n-1)}{1\cdot 2}\left[\frac{n-2}{3} + 1\right]a^{n-2}x^3 + \cdots.\end{aligned}$$

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* A point is often used in place of the sign ×; thus, $1\cdot 2$ is the same as 1×2.

$$\begin{aligned}\text{Or, } (a+x)^{n+1} &= a^{n+1} + (n+1)a^n x + n\left[\frac{n+1}{2}\right]a^{n-1}x^2 \\ &\quad + \frac{n(n-1)}{1\cdot 2}\left[\frac{n+1}{3}\right]a^{n-2}x^3 + \cdots \\ &= a^{n+1} + (n+1)a^n x + \frac{(n+1)n}{1\cdot 2}a^{n-1}x^2 \\ &\quad + \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}a^{n-2}x^3 + \cdots.\end{aligned}$$

It will be observed that this result is in accordance with the laws of § 357; which proves that, if the laws of § 357 hold for any power of $a + x$ whose exponent is a positive integer, they also hold for a power whose exponent is greater by 1.

But the laws have been shown to hold for $(a + x)^4$, and hence they also hold for $(a + x)^5$; and since they hold for $(a + x)^5$, they also hold for $(a + x)^6$; and so on.

Therefore, the laws hold when the exponent is any positive integer, and equation (1) is proved for every positive integral value of n.

Equation (1) is called the *Binomial Theorem.*

Note 1. The above method of proof is known as *Mathematical Induction.*

Note 2. In place of the denominators $1\cdot 2$, $1\cdot 2\cdot 3$, etc., it is usual to write $\underline{|2}$, $\underline{|3}$, etc. The symbol $\underline{|n}$, read "*factorial n*," signifies the product of the natural numbers from 1 to n inclusive.

359. Putting $a = 1$ in equation (1), § 358, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \cdots.$$

EXAMPLES.

360. In expanding expressions by the Binomial Theorem it is convenient to obtain the exponents and coefficients of the terms by aid of the laws of § 357, which have been proved to hold for any positive integral exponent.

1. Expand $(a+x)^5$.

The exponent of a in the first term is 5, in the second term 4, in the third term 3, in the fourth term 2, in the fifth term 1.

The exponent of x in the second term is 1, in the third term 2, in the fourth term 3, in the fifth term 4, in the sixth term 5.

The coefficient of the first term is 1; of the second term, 5.

Multiplying the coefficient of the second term, 5, by 4, the exponent of a in that term, and dividing the result by the exponent of x in the term increased by 1, or 2, we have 10 as the coefficient of the third term; and so on.

Then, $(a+x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$, *Ans.*

Note 1. The coefficients of terms equally distant from the beginning and end of the expansion are equal. Thus the coefficients of the latter half of an expansion may be written out from the first half.

If the second term of the binomial is *negative*, it should be enclosed, sign and all, in a parenthesis before applying the laws. In reducing afterwards, care must be taken to apply the principles of § 186.

2. Expand $(1-x)^6$.

We have,
$$\begin{aligned}(1-x)^6 &= [1+(-x)]^6\\ &= 1^6 + 6\cdot 1^5\cdot(-x) + 15\cdot 1^4\cdot(-x)^2 + 20\cdot 1^3\cdot(-x)^3\\ &\quad + 15\cdot 1^2\cdot(-x)^4 + 6\cdot 1\cdot(-x)^5 + (-x)^6\\ &= 1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6,\ \textit{Ans.}\end{aligned}$$

Note 2. If the first term of the binomial is ***numerical***, it is convenient to write the exponents at first without reduction. The result should afterwards be reduced to its simplest form.

If either term of the binomial has a coefficient or exponent other than unity, it should be enclosed in a parenthesis before applying the laws.

3. Expand $(3m^2 - \sqrt[3]{n})^4$.

$$\begin{aligned}(3m^2 - \sqrt[3]{n})^4 &= [(3m^2) + (-n^{\frac{1}{3}})]^4\\ &= (3m^2)^4 + 4(3m^2)^3(-n^{\frac{1}{3}}) + 6(3m^2)^2(-n^{\frac{1}{3}})^2\\ &\quad + 4(3m^2)(-n^{\frac{1}{3}})^3 + (-n^{\frac{1}{3}})^4\\ &= 81m^8 - 108m^6n^{\frac{1}{3}} + 54m^4n^{\frac{2}{3}} - 12m^2n + n^{\frac{4}{3}},\ \textit{Ans.}\end{aligned}$$

[illegible] simplify until th
[illegible] expansion is indicated

Expand the following:

4. $(x+1)^4$.
5. $(a+x)^6$.
6. $(a-x)^4$.
7. $(m-n)^5$.
8. $(1+x)^7$.
9. $(a-b)^6$.
10. $(a^2+b^3c)^5$.
11. ~~$(x^{2m}+y^{3n})^6$.~~
12. $(2a-1)^4$.
13. $(x+2)^5$.
3 14. $(a-3b)^4$.

4 15. $(1+2m^2)^6$.
5 16. $(1-x)^8$.
17. $(\sqrt[5]{x^4}+x^{-1})^5$.
6 18. $(a^{\frac{1}{2}}-2)^7$.
7 19. $(3+x^3)^5$.
20. $\left(\frac{1}{m^{\frac{3}{4}}}+\sqrt[3]{m^4}\right)^6$.
9 21. $(4a^{\frac{3}{2}}-x^{\frac{1}{3}})^4$.
10 22. $\left(x^{-2}-\frac{y^4}{3}\right)^5$.
23. $(m^3+5x^{-3})^4$.

11 24. $\left(2a^{\frac{2}{3}}+\frac{1}{2a^{\frac{3}{2}}}\right)^4$.
25. $\left(\sqrt[5]{x^2}-\frac{1}{\sqrt[4]{y}}\right)^7$.
26. $(2a^{-\frac{2}{3}}-\sqrt{b})^4$.
27. $\left(\frac{x^{-\frac{3}{4}}}{2}-\sqrt[5]{m^3}\right)^5$.
28. $(\sqrt{a^3}+4\sqrt[3]{a})^4$.
29. $\left(a^{\frac{1}{2}}-\frac{3}{\sqrt[3]{x^4}}\right)^6$.
30. $(2a-3b)^5$.

31. $(a^{\frac{1}{2}}b^{-\frac{1}{3}}+a^{-\frac{1}{2}}b^{\frac{1}{3}})^7$.
32. $\left(3\sqrt{\frac{m}{n}}-2\sqrt{\frac{n}{m}}\right)^4$.

A *trinomial* may be raised to any power by the Binomial Theorem if two of its terms be enclosed in a parenthesis and regarded as a single term.

33. Expand $(x^2-2x-2)^4$.

$$\begin{aligned}(x^2-2x-2)^4 &= [(x^2-2x)+(-2)]^4\\ &= (x^2-2x)^4+4(x^2-2x)^3(-2)+6(x^2-2x)^2(-2)^2\\ &\qquad +4(x^2-2x)(-2)^3+(-2)^4\\ &= x^8-8x^7+24x^6-32x^5+16x^4-8(x^6-6x^5+12x^4-8x^3)\\ &\qquad +24(x^4-4x^3+4x^2)-32(x^2-2x)+16\\ &= x^8-8x^7+16x^6+16x^5-56x^4-32x^3\\ &\qquad +64x^2+64x+16, \textit{ Ans.}\end{aligned}$$

Expand the following:

34. $(1-x+x^2)^4$.
35. $(x^2+x+2)^4$.
36. $(1+3x-x^2)^4$.
37. $(x^2-2x-3)^4$.
38. $(1+x-x^2)^5$.
39. $(x^2-x+2)^5$.

361. *To find the rth or general term in the expansion of* $(a+x)^n$.

The following laws will be found to hold for any term in the expansion of $(a+x)^n$, in equation (1), § 358:

1. The exponent of x is less by 1 than the number of the term.

2. The exponent of a is n minus the exponent of x.

3. The last factor of the numerator is greater by 1 than the exponent of a.

4. The last factor of the denominator is the same as the exponent of x.

Hence, in the rth term, the exponent of x is $r-1$.

The exponent of a is $n-(r-1)$, or $n-r+1$.

The last factor of the numerator is $n-r+2$.

The last factor of the denominator is $r-1$.

Therefore, the rth term of the expansion of $(a+x)^n$ is

$$\frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}a^{n-r+1}x^{r-1}.$$

EXAMPLES.

362. In finding any term of an expansion, it is convenient to obtain the coefficient and the exponents of the terms by aid of the laws of § 361.

1. Find the 8th term of $(3a^{\frac{1}{2}}-b^{-1})^{11}$.

We have, $(3a^{\frac{1}{2}}-b^{-1})^{11}=[(3a^{\frac{1}{2}})+(-b^{-1})]^{11}$.

In this case, $n=11$ and $r=8$.

The exponent of $(-b^{-1})$ is $8-1$, or 7.

The exponent of $(3a^{\frac{1}{2}})$ is $11-7$, or 4.

The first factor of the numerator is 11, and the last factor $4+1$, or 5.

The last factor of the denominator is 7.

Hence, the 8th term $=\dfrac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(3a^{\frac{1}{2}})^4(-b^{-1})^7$

$=330\cdot(81a^2)(-b^{-7})=-26730a^2b^{-7}$, *Ans.*

Note. If the second term of the binomial is negative, it should be enclosed, sign and all, in a parenthesis before applying the laws.

If either term of the binomial has a coefficient or exponent other than unity, it should be enclosed in a parenthesis before applying the laws.

Find the

2. 4th term of $(a+x)^8$.

3. 9th term of $(m+1)^{11}$.

4. 5th term of $(a-b)^9$.

5. 10th term of $(1-x)^{16}$.

6. 9th term of $(m^2-n^3)^{12}$.

7. 7th term of $\left(a^{-3}+\frac{ab}{2}\right)^{10}$.

8. 4th term of $\left(\frac{x^2}{y}-\frac{y^2}{x}\right)^{12}$.

9. 10th term of $(a^m+a^n)^{15}$.

10. 8th term of $(x^{-1}-2y^{\frac{1}{2}})^{13}$.

11. 6th term of $(a^{-\frac{2}{3}}+3x^5)^{10}$

12. 7th term of $\left(a^{\frac{3}{2}}-\frac{1}{4\sqrt{b}}\right)^{13}$.

13. 8th term of $(a^{-\frac{1}{3}}+2\sqrt[3]{x^4})^{11}$.

14. 5th term of $\left(m^{\frac{3}{4}}+\frac{2}{\sqrt[5]{n^3}}\right)^{15}$.

15. 6th term of $\left(x^{-\frac{2}{3}}-\frac{\sqrt[4]{y^3}}{3}\right)^{14}$.

16. Find the middle term of $\left(a^{\frac{1}{4}}-\frac{x^2}{2}\right)^{14}$.

XXXII. UNDETERMINED COEFFICIENTS.

CONVERGENCY AND DIVERGENCY OF SERIES.

363. A **Series** is a succession of terms so related that each may be derived from one or more of the preceding in accordance with some fixed law.

A **Finite Series** is one having a finite number of terms.

An **Infinite Series** is one the number of whose terms is unlimited.

The progressions, in general, are examples of finite series; but in § 347 we considered infinite geometrical series.

364. Infinite series may be developed by Division.

Let it be required, for example, to divide 1 by $1 - x$.

$$\begin{array}{l} 1 - x)\,1\,(1 + x + x^2 + \cdots \\ \quad\;\;\, \underline{1 - x} \\ \qquad\quad x \\ \qquad\quad \underline{x - x^2} \\ \qquad\qquad\;\; x^2 \end{array}$$

Therefore, $\frac{1}{1-x} = 1 + x + x^2 + \cdots$.

Infinite series may also be developed by Evolution (see Exs. 25 to 30, § 195), and by other methods, one of the most important of which will be considered in § 369.

365. A series is said to be *convergent* either when the sum of the first n terms approaches a certain fixed quantity as a limit (§ 292); when n is indefinitely increased; or when the sum of all the terms is equal to a fixed finite quantity.

A series is said to be *divergent* when the sum of the first n terms can be made to numerically exceed any assigned quantity, however great, by taking n sufficiently great.

366. Consider, for example, the infinite series

$$1 + x + x^2 + x^3 + \cdots.$$

I. Suppose $x = x_1$, where x_1 is numerically < 1.

The sum of the first n terms is now

$$1 + x_1 + x_1^2 + \cdots + x_1^{n-1} = \frac{1 - x_1^n}{1 - x_1} \text{ (§ 86).}$$

If n is indefinitely increased, x_1^n decreases indefinitely in absolute value, and approaches the limit 0.

Then the fraction $\frac{1 - x_1^n}{1 - x_1}$ approaches the limit $\frac{1}{1 - x_1}$.

That is, the sum of the first n terms approaches a certain fixed quantity as a limit, when n is indefinitely increased.

Hence, the series is *convergent* when x is numerically < 1.

II. Suppose $x = 1$.

In this case, each term of the series is equal to 1, and the sum of the first n terms is equal to n; and this sum can be made to exceed any assigned quantity, however great, by taking n sufficiently great.

Hence, the series is *divergent* when $x = 1$.

III. Suppose $x = -1$.

In this case, the series takes the form $1 - 1 + 1 - 1 + \cdots$, and the sum of the first n terms is either 1 or 0 according as n is odd or even.

Hence, the series is *neither convergent nor divergent* when $x = -1$.

IV. Suppose $x = x_1$, where x_1 is numerically > 1.

The sum of the first n terms is now

$$1 + x_1 + x_1^2 + \cdots + x_1^{n-1} = \frac{x_1^n - 1}{x_1 - 1} \text{ (§ 86).}$$

By taking n sufficiently great, $\frac{x_1^n - 1}{x_1 - 1}$ can be made to numerically exceed any assigned quantity, however great.

Hence, the series is *divergent* when x is numerically >1.

367. Consider the infinite series

$$1+x+x^2+x^3+\cdots,$$

developed by the fraction $\frac{1}{1-x}$ (§ 364).

Let $x=.1$, in which case the series is convergent (§ 366).

The series now takes the form $1+.1+.01+.001+\cdots$, while the value of the fraction is $\frac{1}{.9}$, or $\frac{10}{9}$.

In this case, however great the number of terms taken, their sum will never exactly equal $\frac{10}{9}$; but it approaches this value as a limit. (See § 347.)

Thus, if an infinite series is *convergent*, the greater the number of terms taken, the more nearly does their sum approach to the value of the expression from which the series was developed.

Again, let $x=10$, in which case the series is divergent.

The series now takes the form $1+10+100+1000+\cdots$, while the value of the fraction is $\frac{1}{1-10}$, or $-\frac{1}{9}$

In this case it is evident that, the greater the number of terms taken, the more does their sum diverge from the value $-\frac{1}{9}$.

Thus, if an infinite series is *divergent*, the greater the number of terms taken, the more does their sum diverge from the value of the expression from which the series was developed.

It follows from the above that *an infinite series cannot be used for the purposes of demonstration, unless it is convergent.*

368. The infinite series

$$a+bx+cx^2+dx^3+\cdots$$

is convergent when $x=0$; for the sum of all the terms is equal to a when $x=0$.

THE THEOREM OF UNDETERMINED COEFFICIENTS.

369. An important method for expanding expressions into series is based on the following theorem:

The Theorem of Undetermined Coefficients.

If the series $A + Bx + Cx^2 + Dx^3 + \cdots$ *is always equal to the series* $A' + B'x + C'x^2 + D'x^3 + \cdots$, *when* x *has any value which makes both series convergent, the coefficients of like powers of* x *in the series will be equal; that is,* $A = A'$, $B = B'$, $C = C'$, *etc.*

For since the equation

$$A + Bx + Cx^2 + Dx^3 + \cdots = A' + B'x + C'x^2 + D'x^3 + \cdots$$

is satisfied when x has any value which makes both members convergent, and since both members are convergent when $x = 0$ (§ 368), it follows that the equation is satisfied when $x = 0$.

Putting $x = 0$, we have $A = A'$.

Subtracting A from the first member of the equation, and its equal A' from the second member, we obtain

$$Bx + Cx^2 + Dx^3 + \cdots = B'x + C'x^2 + D'x^3 + \cdots.$$

Dividing each term by x,

$$B + Cx + Dx^2 + \cdots = B' + C'x + D'x^2 + \cdots.$$

This equation also is satisfied when x has any value which makes both members convergent; and putting $x = 0$, we have $B = B'$.

In like manner, we may prove $C = C'$, $D = D'$, etc.

370. A finite series being always convergent, it follows from the preceding article that if two finite series

$$A + Bx + Cx^2 + \cdots + Kx^n \text{ and } A' + B'x + C'x^2 + \cdots + K'x^n$$

are equal for every value of x, the coefficients of like powers of x in the two series are equal.

EXPANSION OF FRACTIONS INTO SERIES.

371. 1. Expand $\dfrac{2-3x^2-x^3}{1-2x+3x^2}$ in ascending powers of x.

Assume
$$\frac{2-3x^2-x^3}{1-2x+3x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots; \qquad (1)$$
where A, B, C, D, E, etc., are quantities independent of x.

Clearing of fractions, and collecting the terms in the second member involving like powers of x, we have

$$2-3x^2-x^3 = A + \begin{array}{r|r|r|r|l} B & x + \quad C & x^2 + \quad D & x^3 + \quad E & x^4 + \cdots. \\ -2A & -2B & -2C & -2D & \\ & +3A & +3B & +3C & \end{array} \qquad (2)$$

The second member of (1) must express the value of the fraction for every value of x which makes the series convergent (§ 367).

Hence, equation (2) is satisfied when x has any value which makes both members convergent; and by the Theorem of Undetermined Coefficients, the coefficients of like powers of x in the series are equal.

Then, $A = 2$.

$$\begin{array}{rll} B - 2A = 0; & \text{whence, } B = 2A & = 4. \\ C - 2B + 3A = -3; & \text{whence, } C = 2B - 3A - 3 & = -1. \\ D - 2C + 3B = -1; & \text{whence, } D = 2C - 3B - 1 & = -15. \\ E - 2D + 3C = 0; & \text{whence, } E = 2D - 3C & = -27; \text{ etc.} \end{array}$$

Substituting these values in (1), we have

$$\frac{2-3x^2-x^3}{1-2x+3x^2} = 2 + 4x - x^2 - 15x^3 - 27x^4 + \cdots, \quad \textit{Ans.}$$

The result may be verified by division.

Note 1. A vertical line, called a *bar*, is often used in place of a parenthesis.

Thus, $\begin{array}{r|l} +\ B & x \\ -2A & \end{array}$ is equivalent to $(B-2A)x$.

Note 2. The result expresses the value of the given fraction only for such values of x as make the series convergent (§ 367).

If the numerator and denominator contain only *even* powers of x, the operation may be abridged by assuming a series containing only the even powers of x.

Thus, if the fraction were $\frac{2 + 4x^2 - x^4}{1 - 3x^2 + 5x^4}$, we should assume it equal to $A + Bx^2 + Cx^4 + Dx^6 + Ex^8 + \cdots$.

In like manner, if the numerator contains only *odd* powers of x, and the denominator only even powers, we should assume a series containing only the odd powers of x.

If every term of the numerator contains x, we may assume a series commencing with the lowest power of x in the numerator.

If every term of the denominator contains x, we determine by actual division what power of x will occur in the first term of the expansion, and then assume the fraction equal to a series commencing with this power of x, the exponents of x in the succeeding terms increasing by unity as before.

2. Expand $\frac{1}{3x^2 - x^3}$ in ascending powers of x.

Dividing 1 by $3x^2$, the quotient is $\frac{x^{-2}}{3}$; we then assume

$$\frac{1}{3x^2 - x^3} = Ax^{-2} + Bx^{-1} + C + Dx + Ex^2 + \cdots. \qquad (1)$$

Clearing of fractions, we have

$$\begin{array}{ll|l|l|l|l} 1 = 3A & + 3B & x + 3C & x^2 + 3D & x^3 + 3E & x^4 + \cdots. \\ & \quad - A & \quad - B & \quad - C & \quad - D & \end{array}$$

Equating the coefficients of like powers of x,

$$3A = 1.$$

$$3B - A = 0.$$

$$3C - B = 0.$$

$$3D - C = 0.$$

$$3E - D = 0; \text{ etc.}$$

Whence, $A = \frac{1}{3}$, $B = \frac{1}{9}$, $C = \frac{1}{27}$, $D = \frac{1}{81}$, $E = \frac{1}{243}$, etc.

Substituting in (1), we have

$$\frac{1}{3x^2 - x^3} = \frac{x^{-2}}{3} + \frac{x^{-1}}{9} + \frac{1}{27} + \frac{x}{81} + \frac{x^2}{243} + \cdots, \textit{ Ans.}$$

EXAMPLES.

Expand each of the following to five terms in ascending powers of x:

3. $\dfrac{1+5x}{1+x}$.

4. $\dfrac{3-2x}{1-4x}$.

5. $\dfrac{2+7x^2}{1-3x^2}$.

6. $\dfrac{4x-x^3}{2+3x^2}$.

7. $\dfrac{1-x-3x^2}{1-2x-x^2}$.

8. $\dfrac{2x+3x^2-x^3}{1+5x-2x^2}$.

9. $\dfrac{1}{3x^2-5x^3}$.

10. $\dfrac{1-2x}{2-3x+4x^2}$.

11. $\dfrac{1-4x^2+6x^3}{1+2x-x^2}$.

12. $\dfrac{2+x-3x^2}{1-4x+5x^2}$.

13. $\dfrac{1-7x^2-4x^3}{x^3-5x^4-2x^5}$.

14. $\dfrac{3+5x-2x^3}{x^2-3x^3+x^4}$.

15. $\dfrac{x^2-4x^3+2x^5}{2-3x^2-x^3}$.

16. $\dfrac{2-3x^2}{3-2x+x^3}$.

17. $\dfrac{3-4x^3}{2x+x^3-3x^4}$.

EXPANSION OF RADICALS INTO SERIES.

372. 1. Expand $\sqrt{1-x}$ in ascending powers of x.

Assume $\sqrt{1-x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots$. (1)

Squaring both members, we have by the rule of § 187,

$$1 - x = \begin{array}{l|l|l|l|l} A^2 & +B^2 & & +C^2 & \\ & x + 2AB & x^2 + 2AC & x^3 + 2AE & x^4 + \cdots. \\ & & +2AD & +2BD & \\ & & +2BC & & \end{array}$$

Equating the coefficients of like powers of x,

$$A^2 = 1; \quad \text{whence, } A = 1.$$

$$2AB = -1; \text{ whence, } B = -\frac{1}{2A} = -\frac{1}{2}.$$

$$B^2 + 2AC = 0; \quad \text{whence, } C = -\frac{B^2}{2A} = -\frac{1}{8}.$$

$$2AD + 2BC = 0; \quad \text{whence, } D = -\frac{BC}{A} = -\frac{1}{16}.$$

$$C^2 + 2AE + 2BD = 0; \quad \text{whence, } E = -\frac{C^2 + 2BD}{2A} = -\frac{5}{128}; \text{ etc.}$$

Substituting these values in (1), we have

$$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \cdots, \textit{ Ans.}$$

The result may be verified by evolution.

EXAMPLES.

Expand each of the following to five terms in ascending powers of x:

2. $\sqrt{1+4x}$. 4. $\sqrt{1+2x-x^2}$. 6. $\sqrt[3]{1+3x}$.

3. $\sqrt{1-5x}$. 5. $\sqrt{1-x-x^2}$. 7. $\sqrt[3]{1-x+x^2}$.

PARTIAL FRACTIONS.

373. If the denominator of a fraction can be resolved into factors, each of the first degree in x, and the numerator is of a lower degree than the denominator, the Theorem of Undetermined Coefficients enables us to express the given fraction as the sum of two or more *partial fractions,* whose denominators are factors of the given denominator, and whose numerators are independent of x.

374. CASE I. *When no two factors of the denominator are equal.*

1. Separate $\dfrac{19x+1}{(3x-1)(5x+2)}$ into partial fractions.

Assume
$$\frac{19x+1}{(3x-1)(5x+2)}=\frac{A}{3x-1}+\frac{B}{5x+2}, \qquad (1)$$

where A and B are quantities independent of x.

Clearing of fractions, $19x+1=A(5x+2)+B(3x-1)$.

Or, $$19x+1=(5A+3B)x+2A-B. \qquad (2)$$

The second member of (1) must express the value of the given fraction for every value of x.

Hence, equation (2) is satisfied by every value of x; and by § 370, the coefficients of like powers of x in the two members are equal.

That is, $5A+3B=19$,

and $2A-B=1$.

Solving these equations, we obtain $A=2$ and $B=3$.

Substituting in (1), $$\frac{19x+1}{(3x-1)(5x+2)}=\frac{2}{3x-1}+\frac{3}{5x+2}, \textit{ Ans.}$$

The result may be verified by adding the partial fractions.

2. Separate $\frac{x+4}{2x-x^2-x^3}$ into partial fractions.

The factors of $2x-x^2-x^3$ are x, $1-x$, and $2+x$ (§ 284).

Assume then $\frac{x+4}{2x-x^2-x^3}=\frac{A}{x}+\frac{B}{1-x}+\frac{C}{2+x}.$

Clearing of fractions, we have

$$x+4=A(1-x)(2+x)+Bx(2+x)+Cx(1-x).$$

This equation is satisfied by every value of x; it is therefore satisfied when $x=0$.

Putting $x=0$, we have $4=2A$, or $A=2$.

Again, the equation is satisfied when $x=1$.

Putting $x=1$, we have $5=3B$, or $B=\frac{5}{3}$.

The equation is also satisfied when $x=-2$.

Putting $x=-2$, we have $2=-6C$, or $C=-\frac{1}{3}$.

Then,
$$\frac{x+4}{2x-x^2-x^3}=\frac{2}{x}+\frac{\frac{5}{3}}{1-x}+\frac{-\frac{1}{3}}{2+x}$$
$$=\frac{2}{x}+\frac{5}{3(1-x)}-\frac{1}{3(2+x)},\ \textit{Ans.}$$

Note. To find the value of A, in Ex. 2, we give to x *such a value as will make the coefficients of B and C equal to zero;* and we proceed in a similar manner to find the values of B and C.

This method of finding A, B, and C is usually shorter than that used in Ex. 1.

EXAMPLES.

Separate each of the following into partial fractions:

3. $\frac{18x+3}{4x^2-9}$.

4. $\frac{x-2}{5x^2-6x}$.

5. $\frac{x^2-75}{x^3-25x}$.

6. $\frac{38x+5}{6x^2+5x-6}$.

7. $\frac{ax-19a^2}{x^2+4ax-5a^2}$.

8. $\frac{46-5x}{8-18x-5x^2}$.

9. $\frac{x^2+10x-7}{(2x-1)(12x^2-x-6)}$.

10. $\frac{-13x^2+27x+18}{(x^2-2x)(x^2-9)}$.

375. Case II. *When all the factors of the denominator are equal.*

Let it be required to separate $\frac{x^2-11x+26}{(x-3)^3}$ into partial fractions.

Substituting $y+3$ for x, the fraction becomes

$$\frac{(y+3)^2-11(y+3)+26}{y^3}=\frac{y^2-5y+2}{y^3}=\frac{1}{y}-\frac{5}{y^2}+\frac{2}{y^3}.$$

Replacing y by $x-3$, the result takes the form

$$\frac{1}{x-3}-\frac{5}{(x-3)^2}+\frac{2}{(x-3)^3}.$$

This shows that the given fraction can be expressed as the sum of three partial fractions, whose numerators are independent of x, and whose denominators are the powers of $x-3$ beginning with the first and ending with the third.

Similar considerations hold with respect to any example under Case II.; the number of partial fractions in any case being the same as the number of equal factors in the denominator of the given fraction.

EXAMPLES.

376. 1. Separate $\frac{6x+5}{(3x+5)^2}$ into partial fractions.

In accordance with the principle stated in § 375, we assume the given fraction equal to the sum of *two* partial fractions, whose denominators are the powers of $3x+5$ beginning with the first and ending with the *second*.

Thus, $$\frac{6x+5}{(3x+5)^2}=\frac{A}{3x+5}+\frac{B}{(3x+5)^2}.$$

Clearing of fractions, $6x+5=A(3x+5)+B.$

Or, $6x+5=3Ax+5A+B.$

Equating the coefficients of like powers of x,

$$3A=6.$$

$$5A+B=5.$$

Solving these equations, we have $A = 2$ and $B = -5$.

Whence, $$\frac{6x+5}{(3x+5)^2} = \frac{2}{3x+5} - \frac{5}{(3x+5)^2}, \textit{ Ans.}$$

Separate each of the following into partial fractions:

2. $\dfrac{14x-30}{4x^2-12x+9}$.
3. $\dfrac{x^2+4x-1}{(x+5)^3}$.
4. $\dfrac{9x^2-15x-1}{(3x-1)^3}$.
5. $\dfrac{8x^2-19}{(2x-3)^3}$.
6. $\dfrac{10x^2+3x-1}{(5x+2)^3}$.
7. $\dfrac{x^3-3x^2-x}{(x-1)^4}$.
8. $\dfrac{x^3+4x^2+7x+2}{(x+2)^4}$.
9. $\dfrac{18x^3-21x^2+4x}{(3x-2)^4}$.

377. Case III. *When some of the factors of the denominator are equal.*

1. Separate $\dfrac{x^2-4x+3}{x(x+1)^2}$ into partial fractions.

The method in Case III. is a combination of those of Cases I. and II.

We assume $$\frac{x^2-4x+3}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Clearing of fractions, $$x^2-4x+3 = A(x+1)^2 + Bx(x+1) + Cx$$
$$= (A+B)x^2 + (2A+B+C)x + A.$$

Equating the coefficients of like powers of x,

$$A + B = 1.$$
$$2A + B + C = -4.$$
$$A = 3.$$

Solving these equations, we have $A = 3$, $B = -2$, and $C = -8$.

Whence, $$\frac{x^2-4x+3}{x(x+1)^2} = \frac{3}{x} - \frac{2}{x+1} - \frac{8}{(x+1)^2}, \textit{ Ans.}$$

Note. It is impracticable to give an illustrative example for every possible case; but no difficulty will be found in assuming the proper partial fractions if attention is given to the following general rule.

The fraction $\dfrac{X}{(x+a)(x+b)\cdots(x+m)^r\cdots}$ should be put equal to

$$\frac{A}{x+a}+\frac{B}{x+b}+\cdots+\frac{E}{x+m}+\frac{F}{(x+m)^2}+\cdots+\frac{K}{(x+m)^r}+\cdots.$$

Single factors like $x+a$ and $x+b$ having single partial fractions corresponding, arranged as in Case I.; and repeated factors like $(x+m)^r$ having r partial fractions corresponding, arranged as in Case II.

EXAMPLES.

Separate each of the following into partial fractions:

2. $\dfrac{18-5x-3x^2}{x(x-3)^2}.$

3. $\dfrac{8x^3+8x^2-18x-8}{x^4+4x^3}.$

4. $\dfrac{12x^2-11x-38}{(3x-1)(2x+3)^2}.$

5. $\dfrac{2-3x-x^2-2x^3}{x^2(x-1)^2}.$

6. $\dfrac{4-9x-12x^2-2x^3}{x(x+1)(x+2)^2}.$

7. $\dfrac{3x+13}{(2x-3)(8x^2-10x-3)}.$

378. If the degree of the numerator is equal to, or greater than, that of the denominator, the preceding methods are inapplicable.

In such a case, we divide the numerator by the denominator until a remainder is obtained which is of a lower degree than the denominator.

1. Separate $\dfrac{x^3-3x^2-1}{x^2-x}$ into an integral expression and partial fractions.

Dividing x^3-3x^2-1 by x^2-x, the quotient is $x-2$, and the remainder $-2x-1$.

Then, $$\frac{x^3-3x^2-1}{x^2-x}=x-2+\frac{-2x-1}{x^2-x}.$$

Separating $\dfrac{-2x-1}{x^2-x}$ into partial fractions by the method of Case I., the result is $\dfrac{1}{x}-\dfrac{3}{x-1}$.

Whence, $$\frac{x^3-3x^2-1}{x^3-x}=x-2+\frac{1}{x}-\frac{3}{x-1},\ \textit{Ans.}$$

EXAMPLES.

Separate each of the following into an integral expression and two or more partial fractions:

2. $\dfrac{9x^3+9x^2-6}{(x+2)(3x-1)}$.

3. $\dfrac{2x^3-17x^2+44x-29}{(x-2)^3}$.

4. $\dfrac{x^5+2x^3-3x^2+x+3}{x^3(x^2+1)}$.

5. $\dfrac{x^5-2x^3+4x-1}{x^2(x-1)^2}$.

6. $\dfrac{x^6+3x^5+3x^4-10x^2-x+6}{x^4+3x^3}$.

379. If the denominator of a fraction can be resolved into factors partly of the first and partly of the second degree, or all of the second degree, in x, and the numerator is of a lower degree than the denominator, the Theorem of Undetermined Coefficients enables us to express the given fraction as the sum of two or more partial fractions, whose denominators are factors of the given denominator, and whose numerators are independent of x in the case of fractions corresponding to factors of the first degree, and of the form $Ax+B$ in the case of fractions corresponding to factors of the second degree.

1. Separate $\dfrac{1}{x^3+1}$ into partial fractions.

The factors of the denominator are $x+1$ and x^2-x+1.

Assume then $$\frac{1}{x^3+1}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}. \qquad (1)$$

Clearing of fractions, $1=A(x^2-x+1)+(Bx+C)(x+1)$.

Or, $1=(A+B)x^2+(-A+B+C)x+A+C$.

Equating the coefficients of like powers of x,

$$A+B=0.$$

$$-A+B+C=0.$$

$$A+C=1.$$

Solving these equations, we have

$$A=\frac{1}{3},\ B=-\frac{1}{3},\ \text{and}\ C=\frac{2}{3}.$$

Substituting in (1), $\dfrac{1}{x^3+1}=\dfrac{1}{3(x+1)}-\dfrac{x-2}{3(x^2-x+1)}$, *Ans.*

EXAMPLES.

Separate each of the following into partial fractions:

2. $\dfrac{5x^2+1}{x^3+1}$.

3. $\dfrac{x^2+16x-12}{(3x+1)(x^2-x+3)}$.

4. $\dfrac{2x^2+11x-7}{(2x-5)(x^2+2)}$.

5. $\dfrac{3x^3-5x^2+x-3}{x^4-1}$.

6. $\dfrac{12+13x-2x^2}{8x^3-27}$.

7. $\dfrac{2x^3+2x^2+10}{x^4+x^2+1}$.

REVERSION OF SERIES.

380. To *revert* a given series $y=a+bx^m+cx^n+\cdots$ is to express x in the form of a series proceeding in ascending powers of y.

1. Revert the series $y=2x+x^2-2x^3-3x^4+\cdots$.

Assume $\quad x=Ay+By^2+Cy^3+Dy^4+\cdots. \qquad (1)$

Substituting in this the given value of y, and performing the operations indicated, we have

$$\begin{aligned} x = {} & A(2x+x^2-2x^3-3x^4+\cdots) \\ & +B(4x^2+x^4+4x^3-8x^4+\cdots) \\ & +C(8x^3+12x^4+\cdots)+D(16x^4+\cdots)+\cdots. \end{aligned}$$

That is, $x=2Ax+\begin{array}{r|r|r|l} A & x^2-2A & x^3-\ 3A & x^4+\cdots. \\ +4B & +4B & -\ 7B & \\ & +8C & +12C & \\ & & +16D & \end{array}$

Equating the coefficients of like powers of x,

$$2A=1.$$

$$A+4B=0.$$

$$-2A+4B+8C=0.$$

$$-3A-7B+12C+16D=0;\ \text{etc.}$$

Solving these equations,

$$A=\frac{1}{2},\ B=-\frac{1}{8},\ C=\frac{3}{16},\ D=-\frac{13}{128},\ \text{etc.}$$

Substituting in (1), $x=\frac{1}{2}y-\frac{1}{8}y^2+\frac{3}{16}y^3-\frac{13}{128}y^4+\cdots$, *Ans.*

If the even powers of x are wanting in the given series, the operation may be abridged by assuming x equal to a series containing only the *odd* powers of y.

EXAMPLES.

Revert each of the following to four terms:

2. $y=x-x^2+x^3-x^4+\cdots.$

3. $y=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots.$

4. $y=x+2x^2+3x^3+4x^4+\cdots.$

5. $y=x-3x^2+5x^3-7x^4+\cdots.$

6. $y=x-\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}-\frac{x^4}{\underline{|4}}+\cdots.$

7. $y=\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{6}+\frac{x^4}{8}+\cdots.$

8. $y=x+x^3+2x^5+5x^7+\cdots.$

9. $y=x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots.$

XXXIII. THE BINOMIAL THEOREM.

FRACTIONAL AND NEGATIVE EXPONENTS.

381. It was proved in § 359 that, if n is a positive intege

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \cdots \quad (1$$

382. Proof of the Theorem for a Fractional or Negati Exponent.

I. *When the exponent is a positive fraction.*

Let the exponent be $\frac{p}{q}$, where p and q are positive integers.

By § 211, $(1+x)^{\frac{p}{q}} = \sqrt[q]{(1+x)^p} = \sqrt[q]{1 + px + \cdots}$, by (1).

It is evident that a process may be found, analogous t those of §§ 194 and 200, for expanding $\sqrt[q]{1 + px + \cdots}$ i ascending powers of x; and the first term of the result wi evidently be 1.

Assume then, $\sqrt[q]{1 + px + \cdots} = 1 + Mx + Nx^2 + \cdots$. (2

Raising both members to the qth power, we have

$$\begin{aligned} 1 + px + \cdots &= [1 + (Mx + Nx^2 + \cdots)]^q \\ &= 1 + q(Mx + Nx^2 + \cdots) + \cdots, \text{ by (1).} \end{aligned}$$

This equation is satisfied by every value of x which makes both members convergent; and by the Theorem of Undetermined Coefficients (§ 369), the coefficients of x in the two series are equal.

That is, $p = qM$, or $M = \frac{p}{q}$.

Substituting this value in (2), we have

$$(1+x)^{\frac{p}{q}} = 1 + \frac{p}{q}x + \cdots. \quad (3$$

II. *When the exponent is a negative integer or a negative fraction.*

Let the exponent be $-s$, where s is a positive integer or a positive fraction.

By § 214, $(1+x)^{-s} = \dfrac{1}{(1+x)^s} = \dfrac{1}{1+sx+\cdots}$, by (1) or (3).

It is evident that $\dfrac{1}{1+sx+\cdots}$ can be expanded by division in a series proceeding in ascending powers of x; thus,

$$\begin{array}{l} 1+sx+\cdots)\,1\,(1-sx+\cdots \\ \qquad\qquad\quad \underline{1+sx+\cdots} \\ \qquad\qquad\quad -sx-\cdots \end{array}$$

That is, $$(1+x)^{-s} = 1 - sx + \cdots. \qquad (4)$$

From (3) and (4), we observe that, when n is fractional or negative, the form of the expansion is

$$(1+x)^n = 1 + nx + Ax^2 + Bx^3 + \cdots. \qquad (5)$$

Writing $\dfrac{x}{a}$ in place of x, we obtain

$$\left(1+\frac{x}{a}\right)^n = 1 + n\frac{x}{a} + A\frac{x^2}{a^2} + B\frac{x^3}{a^3} + \cdots.$$

Multiplying both members by a^n,

$$(a+x)^n = a^n + na^{n-1}x + Aa^{n-2}x^2 + Ba^{n-3}x^3 + \cdots. \qquad (6)$$

This result is in accordance with the *second, third,* and *fourth* laws of § 357; hence, these three laws hold for fractional or negative values of the exponent.

We will now prove that the *fifth* law of § 357 holds for fractional or negative values of the exponent.

Let P denote the coefficient of x^r, and Q the coefficient of x^{r+1}, in the second member of (5).

Then (5) and (6) may be written

$$(1+x)^n = 1 + nx + \cdots + Px^r + Qx^{r+1} + \cdots, \qquad (7)$$

and $$(a+x)^n = a^n + na^{n-1}x + \cdots$$
$$+ Pa^{n-r}x^r + Qa^{n-r-1}x^{r+1} + \cdots. \qquad (8)$$

In (8) put $a = 1 + y$ and $x = z$; then,

$$(1 + y + z)^n = (1 + y)^n + \cdots + P(1 + y)^{n-r}z^r + \cdots. \quad (9)$$

Again, in (7) put $x = z + y$; then,

$$(1 + z + y)^n = 1 + \cdots + P(z + y)^r + Q(z + y)^{r+1} + \cdots.$$

Expanding the powers of $z + y$ by aid of (8), we have

$$(1 + z + y)^n = 1 + \cdots + P[z^r + rz^{r-1}y + \cdots]$$
$$+ Q[z^{r+1} + (r + 1)z^r y + \cdots] + \cdots. \quad (10)$$

Since the first members of (9) and (10) are identical, their second members must be equal for every value of z which makes both series convergent; and by the theorem of Undetermined Coefficients, the coefficients of z^r in the two series are equal.

Or, $P(1 + y)^{n-r} = P + Q(r + 1)y +$ terms in y^2, y^3, etc.

Expanding the first member by aid of (7), this becomes

$$P[1 + (n - r)y + \cdots] = P + Q(r + 1)y + \cdots.$$

This equation is satisfied by every value of y which makes both members convergent, and hence the coefficients of y in the two series are equal.

That is, $P(n - r) = Q(r + 1)$, or $Q = \dfrac{P(n - r)}{r + 1}$.

But in the second member of (8), $n - r$ is the exponent of a in the term whose coefficient is P, and $r + 1$ is the exponent of x in that term increased by 1.

Hence, the fifth law of § 357 has been proved to hold for fractional or negative values of the exponent.

By aid of the fifth law, the coefficients of the successive terms after the second, in the second member of (8), may be readily found as in § 358; thus,

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n - 1)}{\underline{|2}}a^{n-2}x^2$$
$$+ \frac{n(n - 1)(n - 2)}{\underline{|3}}a^{n-3}x^3 + \cdots. \quad (11)$$

The second member of (11) is an infinite series; for if n is fractional or negative, no one of the quantities $n-1$, $n-2$, etc., can become equal to zero.

The result expresses the value of $(a+x)^n$ only for such values of a and x as make the series convergent (§ 367).

EXAMPLES.

383. In expanding expressions by the Binomial Theorem when the exponent is fractional or negative, the exponents and coefficients of the terms may be obtained by aid of the laws of § 357, which have been proved to hold universally.

If the second term of the binomial is negative, it should be enclosed, sign and all, in a parenthesis before applying the laws; if either term has a coefficient or exponent other than unity, it should be enclosed in a parenthesis before applying the laws.

1. Expand $(a+x)^{\frac{2}{3}}$ to four terms.

The exponent of a in the first term is $\frac{2}{3}$; in the second term, $-\frac{1}{3}$; in the third term, $-\frac{4}{3}$; in the fourth term, $-\frac{7}{3}$; etc.

The exponent of x in the second term is 1; in the third term, 2; in the fourth term, 3; etc.

The coefficient of the first term is 1; of the second term, $\frac{2}{3}$; multiplying the coefficient of the second term, $\frac{2}{3}$, by $-\frac{1}{3}$, the exponent of a in that term, and dividing the result by the exponent of x in the term increased by 1, or 2, we have $-\frac{1}{9}$ as the coefficient of the third term; and so on.

Then, $(a+x)^{\frac{2}{3}} = a^{\frac{2}{3}} + \frac{2}{3}a^{-\frac{1}{3}}x - \frac{1}{9}a^{-\frac{4}{3}}x^2 + \frac{4}{81}a^{-\frac{7}{3}}x^3 - \cdots$, *Ans.*

2. Expand $(1-2x^{-\frac{1}{2}})^{-2}$ to five terms.

$$\begin{aligned}(1-2x^{-\frac{1}{2}})^{-2} &= [1+(-2x^{-\frac{1}{2}})]^{-2}\\ &= 1^{-2} - 2\cdot 1^{-3}\cdot(-2x^{-\frac{1}{2}}) + 3\cdot 1^{-4}\cdot(-2x^{-\frac{1}{2}})^2\\ &\quad - 4\cdot 1^{-5}\cdot(-2x^{-\frac{1}{2}})^3 + 5\cdot 1^{-6}\cdot(-2x^{-\frac{1}{2}})^4 - \cdots\\ &= 1 + 4x^{-\frac{1}{2}} + 12x^{-1} + 32x^{-\frac{3}{2}} + 80x^{-2} + \cdots, \textit{ Ans.}\end{aligned}$$

Or, $$\sqrt[3]{25} = 3 - \frac{2}{3 \cdot 3^2} + \frac{4}{9 \cdot 3^5} - \frac{40}{81 \cdot 3^8} - \cdots.$$

Expressing the value of each fraction approximately to the nearest fifth decimal place, we have

$$\sqrt[3]{25} = 3 - .07407 - .00183 - .00008 - \cdots = 2.92402 \cdots, \textit{ Ans.}$$

Rule.

Separate the given number into two parts, the first of which is the nearest perfect power of the same degree as the required root.

Expand the result by the Binomial Theorem.

Note. If the second term of the binomial is small compared with the first, the terms of the expansion diminish rapidly; but if the second term is large compared with the first, it requires a great many terms to ensure any degree of accuracy.

EXAMPLES.

Find the approximate value of each of the following to five places of decimals:

2. $\sqrt{26}$.
3. $\sqrt{98}$.
4. $\sqrt[3]{9}$.
5. $\sqrt[4]{78}$.
6. $\sqrt[4]{17}$.
7. $\sqrt[5]{29}$.

XXXIV. LOGARITHMS.

386. Every positive number may be expressed, exactly or approximately, as a power of 10.

Thus, $100 = 10^2;\ 13 = 10^{1.1139\cdots};$ etc.

When thus expressed, the corresponding exponent is called its **Logarithm to the Base 10.**

Thus, 2 is the logarithm of 100 to the base 10; a relation which is written $\log_{10} 100 = 2$, or simply $\log 100 = 2$.

387. Logarithms of numbers to the base 10 are called *Common Logarithms,* and, collectively, form the *Common System.*

They are the only ones used for numerical computations.

Any positive number, except unity, may be taken as the base of a system of logarithms; thus, if $a^x = m$, where a and m are positive numbers, then $x = \log_a m$.

Note. A negative number is not considered as having a logarithm.

388. By §§ 213 and 214,

$$\begin{array}{ll} 10^0 = 1, & 10^{-1} = \dfrac{1}{10} = .1, \\ 10^1 = 10, & 10^{-2} = \dfrac{1}{10^2} = .01, \\ 10^2 = 100, & 10^{-3} = \dfrac{1}{10^3} = .001, \text{ etc.} \end{array}$$

Whence by the definition of § 386,

$$\begin{array}{rr} \log 1 = 0, & \log .1 = -1 = 9 - 10, \\ \log 10 = 1, & \log .01 = -2 = 8 - 10, \\ \log 100 = 2, & \log .001 = -3 = 7 - 10, \text{ etc.} \end{array}$$

Note. The second form for log .1, log .01, etc., is preferable in practice. If no base is expressed, the base 10 is understood.

389. It is evident from § 388 that the logarithm of a number greater than 1 is positive, and the logarithm of a number between 0 and 1 negative.

390. If a number is not an exact power of 10, its common logarithm can only be expressed approximately; the integral part of the logarithm is called the *characteristic*, and the decimal part the *mantissa*.

For example, $\log 13 = 1.1139$.

In this case, the characteristic is 1, and the mantissa .1139.

For reasons which will appear hereafter, only the mantissa of the logarithm is given in a table of logarithms of numbers; the characteristic must be found by aid of the rules of §§ 391 and 392.

391. It is evident from § 388 that the logarithm of a number between

1 and 10 is equal to 0 + a decimal;

10 and 100 is equal to 1 + a decimal;

100 and 1000 is equal to 2 + a decimal; etc.

Therefore, the characteristic of the logarithm of a number with *one* figure to the left of the decimal point is 0; with *two* figures to the left of the decimal point is 1; with *three* figures to the left of the decimal point is 2; etc.

Hence, *the characteristic of the logarithm of a number greater than* 1 *is* 1 *less than the number of places to the left of the decimal point.*

For example, the characteristic of log 906328.51 is 5.

392. In like manner, the logarithm of a number between

1 and .1 is equal to 9 + a decimal − 10;

.1 and .01 is equal to 8 + a decimal − 10;

.01 and .001 is equal to 7 + a decimal − 10; etc.

Therefore, the characteristic of the logarithm of a decimal with *no* ciphers between its decimal point and first significant figure is 9, with -10 after the mantissa; of a decimal with *one* cipher between its point and first significant figure is 8, with -10 after the mantissa; of a decimal with *two* ciphers between its point and first significant figure is 7, with -10 after the mantissa; etc.

Hence, *to find the characteristic of the logarithm of a number less than* 1, *subtract the number of ciphers between the decimal point and first significant figure from* 9, *writing* -10 *after the mantissa.*

For example, the characteristic of log .007023 is 7, with -10 written after the mantissa.

Note. Some writers combine the two portions of the characteristic, and write the result as a *negative characteristic* before the mantissa.

Thus, instead of $7.6036-10$, the student will frequently find $\bar{3}.6036$, a minus sign being written over the characteristic to denote that it alone is negative, the mantissa being always positive.

PROPERTIES OF LOGARITHMS.

393. *In any system, the logarithm of* 1 *is* 0.

For by § 213, $a^0=1$; whence by § 387, $\log_a 1=0$.

394. *In any system, the logarithm of the base is* 1.

For, $a^1=a$; whence, $\log_a a=1$.

395. *In any system whose base is greater than* 1, *the logarithm of* 0 *is* $-\infty$.

For if a is greater than 1, $a^{-\infty}=\frac{1}{a^{\infty}}=\frac{1}{\infty}=0$ (§ 295).

Whence by § 387, $\log_a 0=-\infty$.

Note. No literal meaning can be attached to such a result as $\log_a 0=-\infty$; it must be interpreted as follows:

If, in any system whose base is greater than unity, a number approaches the limit 0, its logarithm is negative, and increases without limit in absolute value. (Compare Note to § 296.)

396. *In any system, the logarithm of a product is equal to the sum of the logarithms of its factors.*

Assume the equations

$$\left.\begin{array}{l} a^x = m \\ a^y = n \end{array}\right\}; \text{ whence by § 387, } \left\{\begin{array}{l} x = \log_a m, \\ y = \log_a n. \end{array}\right.$$

Multiplying the assumed equations,

$$a^x \times a^y = mn, \text{ or } a^{x+y} = mn.$$

Whence, $\log_a mn = x + y = \log_a m + \log_a n.$

In like manner, the theorem may be proved for the product of three or more factors.

397. By aid of § 396, the logarithm of a composite number may be found when the logarithms of its factors are known.

1. Given $\log 2 = .3010$ and $\log 3 = .4771$; find $\log 72$.

$$\begin{aligned} \log 72 &= \log (2 \times 2 \times 2 \times 3 \times 3) \\ &= \log 2 + \log 2 + \log 2 + \log 3 + \log 3 \qquad (\S\ 396) \\ &= 3 \times \log 2 + 2 \times \log 3 = .9030 + .9542 = 1.8572, \textit{ Ans.} \end{aligned}$$

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 5 = .6990$, and $\log 7 = .8451$, find:

2. log 35.
3. log 50.
4. log 42.
5. log 75.
6. log 40.
7. log 126.
8. log 196.
9. log 245.
10. log 210.
11. log 625.
12. log 324.
13. log 378.
14. log 405.
15. log 875.
16. log 686.
17. log 1125.
18. log 2625.
19. log 6048.
20. log 12005.
21. log 15876.

398. *In any system, the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.*

Assume the equations

$$\left.\begin{array}{l}a^x = m\\ a^y = n\end{array}\right\};\text{ whence, }\left\{\begin{array}{l}x = \log_a m,\\ y = \log_a n.\end{array}\right.$$

Dividing the assumed equations,

$$\frac{a^x}{a^y} = \frac{m}{n},\text{ or } a^{x-y} = \frac{m}{n}.$$

Whence, $\log_a \frac{m}{n} = x - y = \log_a m - \log_a n.$

399. 1. Given $\log 2 = .3010$; find $\log 5$.

$\log 5 = \log \frac{10}{2} = \log 10 - \log 2$ (§ 398) $= 1 - .3010 = .6990$, *Ans.*

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, and $\log 7 = .8451$, find:

2. $\log \frac{10}{3}$.
3. $\log \frac{7}{4}$.
4. $\log 14\frac{2}{7}$.
5. $\log 45$.
6. $\log \frac{49}{27}$.
7. $\log 225$.
8. $\log \frac{48}{25}$.
9. $\log 6\frac{2}{9}$.
10. $\log 135$.
11. $\log 28\frac{4}{5}$.
12. $\log \frac{200}{9}$.
13. $\log 110\frac{1}{4}$.

400. *In any system, the logarithm of any power of a quantity is equal to the logarithm of the quantity multiplied by the exponent of the power.*

Assume the equation $a^x = m$; whence, $x = \log_a m$.

Raising both members of the assumed equation to the pth power,

$$a^{px} = m^p;\text{ whence, }\log_a m^p = px = p\log_a m.$$

401. *In any system, the logarithm of any root of a quantity is equal to the logarithm of the quantity divided by the index of the root.*

For, $\log_a \sqrt[r]{m} = \log_a (m^{\frac{1}{r}}) = \frac{1}{r}\log_a m$ (§ 400).

402. 1: Given $\log 2 = .3010$; find $\log 2^{\frac{5}{3}}$.

$\log 2^{\frac{5}{3}} = \frac{5}{3} \times \log 2$ (§ 400) $= \frac{5}{3} \times .3010 = .5017$, *Ans.*

Note. To multiply a logarithm by a fraction, multiply first by th numerator, and divide the result by the denominator.

2. Given $\log 3 = .4771$; find $\log \sqrt[8]{3}$.

$$\log \sqrt[8]{3} = \frac{\log 3}{8} = \frac{.4771}{8} = .0596, \textit{ Ans.}$$

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, and $\log 7 = .8451$, fin

3. $\log 3^7$. **7.** $\log 35^5$. **11.** $\log 24^{\frac{2}{7}}$. **15.** $\log \sqrt[5]{1}$

4. $\log 5^6$. **8.** $\log 28^{\frac{1}{3}}$. **12.** $\log \sqrt[7]{3}$. **16.** $\log \sqrt[3]{7}$

5. $\log 2^{\frac{5}{2}}$. **9.** $\log 27^{\frac{1}{5}}$. **13.** $\log \sqrt[6]{5}$. **17.** $\log \sqrt[4]{9}$

6. $\log 7^{\frac{3}{4}}$. **10.** $\log 18^{\frac{5}{6}}$. **14.** $\log \sqrt[9]{7}$. **18.** $\log \sqrt[8]{1}$

19. Find $\log (2^{\frac{1}{3}} \times 3^{\frac{5}{4}})$.

By § 396, $\log (2^{\frac{1}{3}} \times 3^{\frac{5}{4}}) = \log 2^{\frac{1}{3}} + \log 3^{\frac{5}{4}} = \frac{1}{3}\log 2 + \frac{5}{4}\log 3$
$= .1003 + .5964 = .6967$, *Ans.*

Find the values of the following:

20. $\log \sqrt[11]{\frac{7}{2}}$. **22.** $\log (2^{\frac{2}{3}} \times 10^{\frac{1}{2}})$. **24.** $\log \frac{\sqrt[4]{5}}{\sqrt[5]{3}}$. **26.** log

21. $\log \left(\frac{7}{5}\right)^{\frac{1}{4}}$. **23.** $\log 7\sqrt[10]{2}$. **25.** $\log \frac{3^{\frac{4}{5}}}{7^{\frac{1}{4}}}$. **27.** log

403. *To prove the relation*

$$\log_b m = \frac{\log_a m}{\log_a b}.$$

Assume the equations

$$\left.\begin{array}{l} a^x = m \\ b^y = m \end{array}\right\}; \text{ whence, } \left\{\begin{array}{l} x = \log_a m, \\ y = \log_b m. \end{array}\right.$$

From the assumed equations, $a^x = b^y$.

Taking the yth root of both members, $a^{\frac{x}{y}} = b$.

Therefore, $\log_a b = \frac{x}{y}$, or $y = \frac{x}{\log_a b}$.

That is, $\log_b m = \frac{\log_a m}{\log_a b}$.

404. *To prove the relation*

$$\log_b a \times \log_a b = 1.$$

Putting $m = a$ in the result of § 403, we have

$$\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b} \quad (\S\ 394).$$

Whence, $\log_b a \times \log_a b = 1.$

405. *In the common system, the mantissæ of the logarithms of numbers having the same sequence of figures are equal.*

Suppose, for example, that $\log 3.053 = .4847$.

$$\begin{aligned}\text{Then, } \log 305.3 &= \log (100 \times 3.053) = \log 100 + \log 3.053\\ &= 2 + .4847 = 2.4847;\\ \log .03053 &= \log (.01 \times 3.053) = \log .01 + \log 3.053\\ &= 8 - 10 + .4847 = 8.4847 - 10;\ \text{etc.}\end{aligned}$$

It is evident from the above that, if a number be multiplied or divided by any integral power of 10, producing another number with the same sequence of figures, the mantissæ of their logarithms will be equal.

The reason will now be seen for the statement made in § 390, that only the mantissæ are given in a table of logarithms of numbers.

For, to find the logarithm of any number, we have only to take from the table the mantissa corresponding to its sequence of figures, and the characteristic may then be prefixed in accordance with the rules of §§ 391 and 392.

Thus, if $\log 3.053 = .4847$, then

$$\begin{aligned}\log 30.53 &= 1.4847, & \log .3053 &= 9.4847 - 10,\\ \log 305.3 &= 2.4847, & \log .03053 &= 8.4847 - 10,\\ \log 3053. &= 3.4847, & \log .003053 &= 7.4847 - 10,\ \text{etc.}\end{aligned}$$

This property is only enjoyed by the common system of logarithms, and constitutes its superiority over others for the purposes of numerical computation.

406. 1. Given $\log 2 = .3010$, $\log 3 = .4771$; find $\log .00432$

We have, $\log 432 = \log (2^4 \times 3^3) = 4 \log 2 + 3 \log 3 = 2.6353$.

Then by § 405, the *mantissa* of the result is .6353.

Whence by § 392, $\log .00432 = 7.6353 - 10$, *Ans.*

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, and $\log 7 = .8451$, find:

2. $\log 2.8$.
3. $\log 11.2$.
4. $\log .63$.
5. $\log .098$.
6. $\log 32.4$.
7. $\log .00375$.
8. $\log 6750$.
9. $\log .0392$.
10. $\log .000343$.
11. $\log .875$.
12. $\log 2.592$.
13. $\log 274.4$.
14. $\log (3.5)^6$.
15. $\log \sqrt[7]{6.4}$.
16. $\log (12.6)^{\frac{1}{3}}$.

USE OF THE TABLE.

407. The table (pages 348 and 349) gives the mantissæ of the logarithms of all integers from 100 to 1000, calculated to four places of decimals.

408. *To find the logarithm of a number of three figures.*

Look in the column headed "No." for the first two significant figures of the given number.

Then the mantissa required will be found in the corresponding horizontal line, in the vertical column headed by the third figure of the number.

Finally, prefix the characteristic in accordance with the rules of §§ 391 or 392.

For example, $\log 168 = 2.2253$;

$\log .344 = 9.5366 - 10$; etc.

409. For a number consisting of one or two significant figures, the column headed 0 may be used.

Thus, let it be required to find $\log 83$ and $\log 9$.

By § 405, log 83 has the same mantissa as log 830, and log 9 the same mantissa as log 900.

Hence, $\log 83 = 1.9191$, and $\log 9 = 0.9542$.

410. *To find the logarithm of a number of more than three figures.*

Let it be required to find the logarithm of 327.6.

From the table, $\log 327 = 2.5145$,

and $\log 328 = 2.5159$.

That is, an increase of one unit in the number produces an increase of .0014 in the logarithm.

Therefore, an increase of .6 of a unit in the number will produce an increase of $.6 \times .0014$ in the logarithm, or .0008 to the nearest fourth decimal place.

Whence, $\log 327.6 = 2.5145 + .0008 = 2.5153$.

Note. The difference between any mantissa in the table and the mantissa of the next higher number of three figures is called the *tabular difference.* The subtraction may be performed mentally.

The following rule is derived from the above:

Find from the table the mantissa of the first three significant figures, and the tabular difference.

Multiply the latter by the remaining figures of the number, with a decimal point before them.

Add the result to the mantissa of the first three figures, and prefix the proper characteristic.

EXAMPLES.

411. 1. Find log .021508.

Tabular difference = 21	Mantissa of 215 = 3324
.08	2
Correction = 1.68 = 2, nearly.	3326
	Result, 8.3326 − 10.

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

Find the logarithms of the following:

2. 53.
3. 2.6.
4. 871.
5. .689.
6. 1068.
7. 82.95.
8. .9616.
9. .007254.
10. 7.803.
11. .0003787.
12. 253.07.
13. .91873.
14. 4072.6.
15. .0064685.
16. .013592.
17. 4.0354.

412. *To find the number corresponding to a logarithm.*

1. Required the number whose logarithm is 1.6571.

Find in the table the mantissa 6571.

In the corresponding line, in the column headed "No.," we find 45, the first two figures of the required number, and at the head of the column we find 4, the third figure.

Since the characteristic is 1, there must be two places to the left of the decimal point (§ 391).

Hence, the number corresponding to 1.6571 is 45.4.

2. Required the number whose logarithm is 2.3934.

We find in the table the mantissæ 3927 and 3945, whose corresponding numbers are 247 and 248, respectively.

That is, an increase of 18 in the mantissa produces an increase of one unit in the number corresponding.

Therefore, an increase of 7 in the mantissa will produce an increase of $\frac{7}{18}$ of a unit in the number, or .39, nearly.

Hence, the number corresponding is 247 + .39, or 247.39.

The following rule is derived from the above:

Find from the table the next less mantissa, the three figures corresponding, and the tabular difference.

Subtract the next less from the given mantissa, and divide the remainder by the tabular difference.

Annex the quotient to the first three figures of the number, and point off the result.

Note. The rules for pointing off are the reverse of the rules for characteristic given in §§ 391 and 392.

I. *If* – 10 *is not written after the mantissa, add* 1 *to the characteristic, giving the number of places to the left of the decimal point.*

II. *If* – 10 *is written after the mantissa, subtract the positive part of the characteristic from* 9, *giving the number of ciphers to be placed between the decimal point and first significant figure.*

EXAMPLES.

413. 1. Find the number whose logarithm is 8.5264 – 10.

5264
Next less mantissa = 5263; three figures corresponding, 336.
Tabular difference, 13)1.000(.077 = .08, nearly.
91
90

According to the rule of § 412, there will be one cipher between the decimal point and first significant figure.

Hence, the number corresponding = .033608, *Ans.*

Find the numbers corresponding to the following logarithms:

2. 0.3075.
3. 8.7284 – 10.
4. 1.8079.
5. 3.3565.
6. 2.6639.
7. 9.9108 – 10.
8. 7.6899 – 10.
9. 0.8744.
10. 8.9645 – 10.
11. 1.8077.
12. 7.5862 – 10.
13. 9.7043 – 10.
14. 2.5524.
15. 4.2306.
16. 6.2998 – 10.

APPLICATIONS.

414. The approximate value of an arithmetical quantity, in which the operations indicated involve only multiplication, division, involution, or evolution, may be conveniently found by logarithms.

The utility of the process consists in the fact that addition takes the place of multiplication, subtraction of division, multiplication of involution, and division of evolution.

Note. In computations with four-place logarithms, the results cannot usually be depended upon to more than *four* significant figures.

415. 1. Find the value of $.0631 \times 7.208 \times .51272$.

By § 396,

$\log(.0631 \times 7.208 \times .51272) = \log .0631 + \log 7.208 + \log .51272.$

$$\begin{aligned} \log .0631 &= 8.8000 - 10 \\ \log 7.208 &= 0.8578 \\ \log .51272 &= \underline{9.7099 - 10} \end{aligned}$$

Adding, log of result $= 19.3677 - 20 = 9.3677 - 10$. (See Note 1.)

Number corresponding to $9.3677 - 10 = .2332$, *Ans.*

Note 1. If the sum is a negative logarithm, it should be written in such a form that the negative portion of the characteristic may be -10.

Thus, $19.3677 - 20$ is written in the form $9.3677 - 10$.

2. Find the value of $\dfrac{336.8}{7984}$.

By § 398, $\log \dfrac{336.8}{7984} = \log 336.8 - \log 7984.$

$$\begin{aligned} \log 336.8 &= 12.5273 - 10 \qquad \text{(See Note 2.)} \\ \log 7984 &= \underline{3.9022 } \end{aligned}$$

Subtracting, log of result $= 8.6251 - 10$

Number corresponding $= .04218$, *Ans.*

Note 2. To subtract a greater logarithm from a less, or to subtract a negative logarithm from a positive, increase the characteristic of the minuend by 10, writing -10 after the mantissa to compensate.

Thus, to subtract 3.9022 from 2.5273, write the minuend in the form $12.5273 - 10$; subtracting 3.9022 from this, the result is $8.6251 - 10$.

3. Find the value of $(.07396)^5$.

By § 400, $\log(.07396)^5 = 5 \times \log .07396.$

$$\begin{aligned} \log .07396 &= 8.8690 - 10 \\ &\underline{5} \\ &\ 44.3450 - 50 = 4.3450 - 10 \qquad \text{(See Note 1.)} \end{aligned}$$

Number corresponding $= .000002213$, *Ans.*

4. Find the value of $\sqrt[3]{.035063}$.

By § 401,
$$\log \sqrt[3]{.035063} = \frac{1}{3} \log .035063.$$
$$\begin{array}{r} \log .035063 = 8.5449 - 10 \\ 3\underline{)28.5449 - 30} \\ 9.5150 - 10 \end{array}$$
(See Note 3.)

Number corresponding = .3274, *Ans.*

Note 3. To divide a negative logarithm, write it in such a form that the negative portion of the characteristic may be exactly divisible by the divisor, with − 10 as the quotient.

Thus, to divide 8.5449 − 10 by 3, we write the logarithm in the form 28.5449 − 30. Dividing this by 3, the quotient is 9.5150 − 10.

ARITHMETICAL COMPLEMENT.

416. The *Arithmetical Complement* of the logarithm of a number, or, briefly, the *Cologarithm* of the number, is the logarithm of the reciprocal of that number.

Thus,
$$\text{colog}\, 409 = \log \frac{1}{409} = \log 1 - \log 409.$$
$$\begin{array}{rl} \log 1 = & 10 \quad\quad -10 \\ \log 409 = & \underline{2.6117 \quad\quad\quad} \\ \therefore \text{colog}\, 409 = & 7.3883 - 10. \end{array}$$
(Note 2, § 415.)

Again,
$$\text{colog}\, .067 = \log \frac{1}{.067} = \log 1 - \log .067.$$
$$\begin{array}{rl} \log 1 = & 10 \quad\quad -10 \\ \log .067 = & \underline{8.8261 - 10} \\ \therefore \text{colog}\, .067 = & 1.1739. \end{array}$$

It follows from the above that *the cologarithm of a number may be found by subtracting its logarithm from* 10 − 10.

Note. The cologarithm may be obtained by subtracting the last *significant* figure of the logarithm from 10 and each of the others from 9, − 10 being written after the result in the case of a positive logarithm.

417. *Example.* Find the value of $\frac{.51384}{8.709 \times .0946}$.

$$\begin{aligned}\log\frac{.51384}{8.709 \times .0946} &= \log\left(.51384 \times \frac{1}{8.709} \times \frac{1}{.0946}\right)\\ &= \log .51384 + \log\frac{1}{8.709} + \log\frac{1}{.0946}\\ &= \log .51384 + \text{colog}\, 8.709 + \text{colog}\, .0946.\end{aligned}$$

$$\begin{aligned}\log .51384 &= 9.7109 - 10\\ \text{colog}\, 8.709 &= 9.0601 - 10\\ \text{colog}\, .0946 &= 1.0241\\ \hline & 9.7951 - 10 = \log .6239,\ \textit{Ans.}\end{aligned}$$

It is evident from the above example that the logarithm of a fraction is equal to the logarithm of the numerator *plus* the cologarithm of the denominator.

Or in general, to find the logarithm of a fraction whose terms are composed of factors,

Add together the logarithms of the factors of the numerator, and the cologarithms of the factors of the denominator.

Note. The value of the above fraction may be found without using cologarithms, by the following formula:

$$\begin{aligned}\log\frac{.51384}{8.709 \times .0946} &= \log .51384 - \log(8.709 \times .0946)\\ &= \log .51384 - (\log 8.709 + \log .0946).\end{aligned}$$

The advantage in the use of cologarithms is that the written work of computation is exhibited in a more compact form.

EXAMPLES.

Note. A *negative* quantity has no common logarithm (§ 387, Note). If such quantities occur in computation, they should be treated as if they were positive, and the *sign* of the result determined irrespective of the logarithmic work.

Thus, in Ex. 3, § 418, the value of 847.5 × (– 2.2807) is obtained by finding the value of 847.5 × 2.2807, and putting a negative sign before the result. See also Ex. 34.

418. Find by logarithms the values of the following:

1. 3.142×60.39.
2. $541.21 \times .01523$.
3. $847.5 \times (-2.2807)$.
4. $(-4.3918) \times (-.070376)$.
5. $.93653 \times .0031785$.
6. $(-.00017435) \times 69.571$.
7. $\dfrac{486.7}{76.51}$.
8. $\dfrac{1.0547}{34.946}$.
9. $\dfrac{.5394}{-.09216}$.
10. $\dfrac{2.708}{.0086819}$.
11. $\dfrac{9563.2}{42712}$.
12. $\dfrac{-.00006802}{.0071264}$.
13. $\dfrac{3.8961 \times .6945}{4694 \times .00457}$.
14. $\dfrac{718 \times (-.02415)}{(-.5157) \times 1420.6}$.
15. $\dfrac{(-.87028) \times 3.74}{(-.06589) \times (-42.318)}$.
16. $\dfrac{.09213 \times (-73.36)}{.832 \times 2808.7}$.
17. $(7.795)^4$.
18. $(.8328)^7$.
19. $(-25.14)^3$.
20. $(.03512)^2$.
21. $10^{\frac{5}{2}}$.
22. $(.7)^{\frac{3}{4}}$.
23. $(-964)^{\frac{4}{5}}$.
24. $(.00105)^{\frac{5}{3}}$.
25. $\sqrt{5}$.
26. $\sqrt[5]{2}$.
27. $\sqrt[9]{-9}$.
28. $\sqrt[8]{100}$.
29. $\sqrt[4]{.1994}$.
30. $\sqrt[6]{.07256}$.
31. $\sqrt[3]{.002613}$.
32. $\sqrt[7]{-.00095173}$.

33. Find the value of $\dfrac{2\sqrt[3]{5}}{3^{\frac{5}{6}}}$.

$$\log \frac{2\sqrt[3]{5}}{3^{\frac{5}{6}}} = \log 2 + \log \sqrt[3]{5} + \text{colog}\, 3^{\frac{5}{6}} \quad (\S\ 417)$$

$$= \log 2 + \frac{1}{3} \log 5 + \frac{5}{6} \text{colog}\ 3.$$

$$\log 2 = .3010$$

$$\log 5 = .6990; \quad \text{divide by } 3 = .2330$$

$$\text{colog}\ 3 = 9.5229 - 10; \quad \text{multiply by } \frac{5}{6} = \underline{9.6024 - 10}$$

$$.1364 = \log 1.369, \ \textit{Ans.}$$

34. Find the value of $\sqrt[3]{\dfrac{-.03296}{7.962}}$.

$$\log\sqrt[3]{\frac{.03296}{7.962}}=\frac{1}{3}\log\frac{.03296}{7.962}=\frac{1}{3}(\log .03296-\log 7.962).$$

$$\begin{aligned}\log .03296 &= 8.5180-10\\ \log\ 7.962 &= 0.9010\end{aligned}$$

$$3\overline{)27.6170-30}$$

$$9.2057-10=\log .1606.$$

Result, $-.1606$.

Find the values of the following:

35. $4^{\frac{4}{3}}\times 7^{\frac{4}{5}}$.

36. $\dfrac{3^{\frac{5}{4}}}{8^{\frac{2}{9}}}$.

37. $\sqrt[10]{\dfrac{79}{46}}$.

38. $\dfrac{(.01)^{\frac{3}{2}}}{\sqrt[5]{7}}$.

39. $\dfrac{\sqrt{.08}}{(-10)^{\frac{3}{5}}}$.

40. $\left(-\dfrac{4400}{6937}\right)^{\frac{2}{5}}$.

41. $\sqrt{\dfrac{276.8}{940}}$.

42. $\dfrac{5^{\frac{7}{4}}}{\sqrt[3]{-.1}}$.

43. $\dfrac{-\sqrt[4]{1000}}{(-.6)^{\frac{4}{5}}}$.

44. $\sqrt[6]{\dfrac{3}{5}}\div\sqrt[5]{\dfrac{7}{8}}$.

45. $\sqrt[3]{3}\times\sqrt[5]{5}\times\sqrt[7]{7}$.

46. $\left(\dfrac{76\times .0592}{1.307}\right)^{\frac{3}{4}}$.

47. $\sqrt[3]{-\dfrac{7.543}{31\times .414}}$.

48. $\dfrac{\sqrt[4]{.0009657}}{\sqrt[3]{.004978}}$.

49. $\dfrac{-(.25691)^{\frac{6}{5}}}{(-.83457)^{\frac{7}{5}}}$.

50. $(25.467)^{10}\times(-.062)^{12}$.

51. $\sqrt[8]{5106.5\times .00003109}$.

52. $(83.74\times .009433)^{\frac{2}{7}}$.

53. $(4.8671)^{\frac{7}{2}}\times(.17543)^{\frac{1}{5}}$.

54. $\dfrac{\sqrt{3.928}\times\sqrt[4]{65.47}}{\sqrt[6]{721.32}}$.

55. $\dfrac{(.573)^{\frac{3}{5}}}{8693.8\times\sqrt[4]{.03307}}$.

56. $\dfrac{(-.0001916)^{\frac{2}{3}}\times\sqrt{68.1}}{-.27556}$.

57. $\sqrt{.374}\times\sqrt[4]{.05286}\times\sqrt[9]{.0078359}$.

58. $\dfrac{38.014}{\sqrt[5]{.04142}\times(-.947^{\frac{3}{4}})}$.

EXPONENTIAL EQUATIONS.

419. An **Exponential Equation** is an equation of the form $a^x = b$.

To solve an equation of this form, take the logarithms of both members; the result will be an equation which can be solved by ordinary algebraic methods.

1. Given $31^x = 23$; find the value of x.

Taking the logarithms of both members,

$$\log(31^x) = \log 23.$$

Whence, $$x \log 31 = \log 23 \text{ (§ 400)}.$$

Therefore, $$x = \frac{\log 23}{\log 31} = \frac{1.3617}{1.4914} = .91303, \textit{ Ans.}$$

2. Given $.2^x = 3$; find the value of x.

Taking the logarithms of both members,

$$x \log .2 = \log 3.$$

Whence, $$x = \frac{\log 3}{\log .2} = \frac{.4771}{9.3010 - 10} = \frac{.4771}{-.6990} = -.6825, \textit{ Ans.}$$

EXAMPLES.

Solve the following equations:

3. $332^x = 5.17$. 5. $.0158^x = .008295$. 7. $a^x = b^{2x}c^5$.

4. $.416^x = 6.72$. 6. $5.336^x = .744$. 8. $m^4 a^{\frac{3}{x}} = n$.

9. $7^{2x-3} = .02041$. 10. $.8^{x^2-3x} = .4096$.

11. Given a, r, and l; derive the formula for n (§ 346).

12. Given a, r, and S; derive the formula for n.

13. Given a, l, and S; derive the formula for n.

14. Given r, l, and S; derive the formula for n.

420. 1. Find the logarithm of .3 to the base 7.

By § 403, $$\log_7 .3 = \frac{\log_{10} .3}{\log_{10} 7} = \frac{9.4771 - 10}{.8451} = \frac{-.5229}{.8451} = -.6187, \textit{ Ans.}$$

EXAMPLES.

Find the values of the following:

2. $\log_2 13$.

3. $\log_5 .9$.

4. $\log_{.68} 2.9$.

5. $\log_{.34} .076$.

6. $\log_{1.6} .838$.

7. $\log_{83} 5.2$.

Examples like the above may be solved by inspection if the number can be expressed as an exact power of the base.

8. Find the logarithm of 128 to the base 16.

Let $\log_{16} 128 = x$; then by § 387, $16^x = 128$.

That is, $(2^4)^x = 2^7$, or $2^{4x} = 2^7$.

Whence by inspection, $4x = 7$, or $x = \frac{7}{4}$.

Therefore, $\log_{16} 128 = \frac{7}{4}$, *Ans.*

9. Find the logarithm of 81 to the base 3.

10. Find the logarithm of 32 to the base 8.

11. Find the logarithm of $\frac{1}{3}$ to the base 27.

12. Find the logarithm of $\frac{1}{64}$ to the base $\frac{1}{32}$.

APPENDIX.

GRAPHS.*

Note. Sections 1–13 may be studied when the class has reached Article 168 in this book; the study of sections 14–17 may precede that of Article 268, and the later sections may be completed when the class has reached Article 281.

1. A **Graph** is the representation by means of a line or curve of some set of facts. It has a wide application, and is used to show the fluctuations in the price of wheat, the variations in temperature for a day, the death rate at various ages, or the essential facts of an algebraical equation.

2. An equation of the first degree in two variables,

$$3x + 4y = 12.$$

We say in algebra that such an equation is indeterminate, for we can get an indefinite number of values of x and y that will satisfy it. To get a solution we need only to assign arbitrarily to x a value and then solve for y. The following is a set of solutions:

$x = 0$	$y = 3$
$x = 1$	$y = 2\frac{1}{4}$
$x = 2$	$y = 1\frac{1}{2}$
$x = 3$	$y = \frac{3}{4}$
$x = 4$	$y = 0$
$x = 5$	$y = -\frac{3}{4}$

The list could be extended indefinitely.

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3. **Axes.** Two lines intersecting at right angles, as in Fig. 1, are called the **Axes of Coördinates**. OX, the horizontal one, is the **X-axis**, or **Axis** of Abscissas; OY, the vertical one, is the **Y-axis**, or **Axis** of Ordinates. O, the intersection of the axes, is the **origin**.

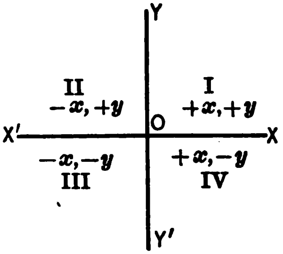

FIG. 1.

4. **Quadrants.** The axes divide the plane into four parts, called quadrants. These quadrants are numbered from I to IV, as in Fig. 1.

5. **Coördinates.** A point is located when its distance and direction from each of the axes is known. The distance from OY is the x-distance, or **abscissa**. The distance from OX is the y-distance, or **ordinate**. The two distances constitute the **Coördinates** of the point. A point is denoted by the symbol (x, y) where x is the abscissa and y the ordinate.

6. **Convention as to Signs.** In the representation of points distances to the right of the Y-axis are positive, to the left negative. Distances above the X-axis are positive, those below negative. An x in the first or fourth quadrant is +, in the second or third it is −. A y in the first or second quadrant is +, in the third or fourth it is −. These are indicated in Fig. 1.

7. **Plotting Points.** To locate the point $P_1(3, 4)$, we measure 3 units to the right of OY and then measure 4 units up from OX. (See Fig. 2.) The point $P_2(-2, 3)$ is 2 units to the left of OY and 3 units above OX. The point $P_3(-3, -1)$ is 3 units to the left of OY and 1 unit below OX. The point $(+2, -3)$ is 2 units to the right of OY and 3 units below OX.

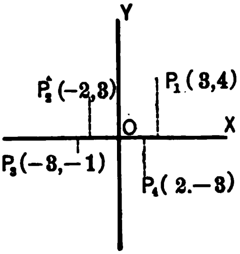

FIG. 2.

Exercises. Plot the following points.

$(2, -4)$, $(-1, -5)$, $(-6, 3)$, $(4, 4)$, $(-3, -5)$,
$(2, -2)$, $(-2, 2)$, $(4, -1)$, $(-5, -1)$, $(4, -3)$,
$(0, 5)$, $(0, -4)$, $(3, 0)$, $(-4, 0)$, $(0, 0)$.

8. The Graph of $3x + 4y = 12$.

In section 2 we wrote down six solutions to this equation. In the notation of section 5, we can now write these solutions as the points $(0, 3)$, $(1, 2\frac{1}{4})$, $(2, 1\frac{1}{2})$, $(3, \frac{3}{4})$, $(4, 0)$, and $(5, -\frac{3}{4})$. These points may be plotted as in Fig. 3. It will now be noticed that these points are in a straight line. The line is called the **graph** of the **equation.** If the line be produced indefinitely and the x and y of any point found by measurement from the graph, the values thus found will satisfy the equation. For example, if we select the point Q, we find that its x, OM, is -4 and its y, MQ is $+6$. These values satisfy the equation $3x + 4y = 12$, for $3(-4) + 4(6) = 12$.

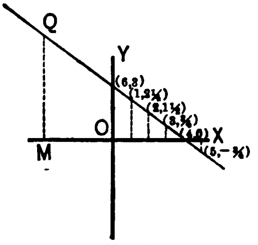

FIG. 3.

9. An equation of the first degree in two variables always has a *straight line* for its *graph*.

$ax + by = c$ is a general linear equation in x and y.

A set of solutions is as follows:

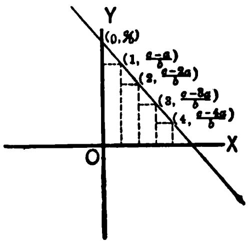

FIG. 4.

$$x = 0, \quad y = \frac{c}{b}$$

$$x = 1, \quad y = \frac{c - a}{b}$$

$$x = 2, \quad y = \frac{c - 2a}{b}$$

$$x = 3, \quad y = \frac{c - 3a}{b}$$

$$x = 4, \quad y = \frac{c - 4a}{b}$$

etc., etc.

We see that a change of 1 in the value of x makes a change $-\frac{a}{b}$ in y. If these points were plotted, they would appear (Fig. 4) very much like the side view of a uniform straight stairway, in which the width of the steps is 1, and the height $\frac{a}{b}$. The points are readily seen to be in a straight line.

A careful plotting of the following equations will do much to convince the student of the truth of the above.

Exercises. Get five or more solutions to each of the following, plot the points, and draw the graph of the equation:

$$x + y = 5, \qquad 2x + 3y = 6, \qquad 3x - 2y = 12.$$

$$4x + y = 7, \qquad x + 4y = 10, \qquad 2x - y = 5.$$

10. A shorter way of getting the graph.

Since the linear equation always represents a straight line, we can draw its graph if we know two points upon it. In general the two points most easily determined are those where the graph cuts the coördinate axes. The point on the X-axis is found by putting $y = 0$ and solving for x. The point on the Y-axis is found by putting $x = 0$ and solving for y.

Example. $2x - 5y = 10$.

If $\qquad y = 0 \qquad x = 5.$

and if $\qquad x = 0 \qquad y = -2.$

The required graph cuts the X-axis at (5, 0) and the Y-axis at (0, $-$2). Plotting these two points the line is easily drawn as in Fig. 5.

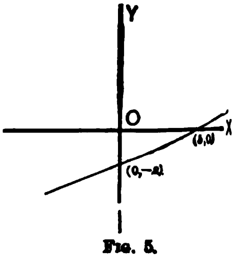

Fig. 5.

11. Simultaneous equations.

$$\begin{matrix}(1)\\(2)\end{matrix}\left\{\begin{aligned}5x+4y&=22\\3x+\ \ y&=\ \ 9\end{aligned}\right\}\text{(problem 3, page 141).}$$

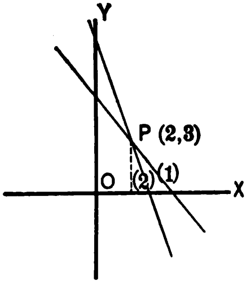

FIG. 6.

Draw the graphs of these two equations on the same diagram as in Fig. 6. It is found that the two lines intersect at a point P whose coördinates are (2, 3). The x and y (2 and 3) of this point is the solution of the equations.

12. Two *simultaneous linear equations* in x and y have but *one* solution. Each equation represents a straight line. The solution is the point common to both lines; that is, the intersection of the lines. Two straight lines can only intersect in one point, so there is but one solution.

Exercises. Solve by drawing graphs and measuring the coördinates of the point of intersection:

Page 141, Examples 5, 8, 13, 18.
Page 142, Examples 2, 4, 7, 17.
Page 143, Examples 2, 5, 9, 15.

13.

$$\begin{matrix}(1)\\(2)\end{matrix}\left\{\begin{aligned}x+\ \ y&=1\\2x+2y&=7\end{aligned}\right\}$$

If we undertake to solve the above equations, we encounter a difficulty. We find that we can not eliminate x without also eliminating y at the same time.

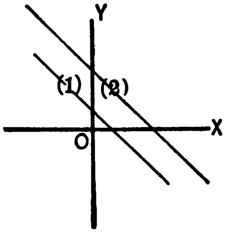

FIG. 7.

If we draw the graphs of these equations, we find they are represented as in Fig. 7. The graphs show at once where the difficulty is. The lines are parallel and so do not intersect at all. In the language of mathematics, theyintersect at infinity, which is just another way of saying they never intersect.

Exercises. Draw the graphs of the following sets of equations:

$$\begin{cases} 2x + y = 4 \\ 4x + 2y = 10 \end{cases} \qquad \begin{cases} 2x + 3y = 12 \\ 3x - 2y = 5 \end{cases}$$

$$\begin{cases} x + 2y = 6 \\ y = 3 \end{cases} \qquad \begin{cases} x = 5 \\ 2x - y = 7 \end{cases}$$

THE GRAPH OF THE QUADRATIC.

14. In order to get the graph of the quadratic we introduce y, putting the equation in the form $ax^2 + bx + c = y$. It is readily seen that when $y = 0$ this equation becomes the general quadratic in x.

15. Graph of $x^2 - 2x - 3 = 0$.

We write $x^2 - 2x - 3 = y$. Solving this for x in the usual way, we get

$$x = 1 \pm \sqrt{4 + y}.$$

The following list of values for y and x are readily found:

1.	$y = 0$	$x = 3$	and	-1
2.	$y = -1$	$x = 2.7$	"	$-.7$
3.	$y = -2$	$x = 2.4$	"	-4
4.	$y = -3$	$x = 2$	"	0
5.	$y = -4$	$x = 1$	"	1
6.	$y = 1$	$x = 3.2$	"	-1.2
7.	$y = 2$	$x = 3.4$	"	-1.4
8.	$y = 3$	$x = 3.6$	"	-1.6
9.	$y = 4$	$x = 3.8$	"	-1.8
10.	$y = 5$	$x = 4$	"	-2
11.	$y = 12$	$x = 5$	"	-3
	etc.	etc.		

Plotting these points carefully and connecting them by a smooth curve, we get the result shown in Fig. 8. It is seen that the graph in this case is a curve, and that it cuts the axis of x in two points. These points are at distances of 3 and -1 from the origin. 3 and -1 are the two roots of the quadratic $x^2 - 2x - 3 = 0$.

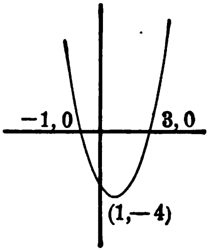

Fig. 8.

16. A quadratic always represents a curve that can be cut in two places by one straight line.

Draw the graphs of the following equations:

1. $x^2 + 6x = 7$.
2. $x^2 + x = 30$.
3. $4x^2 + 7x = 2$.
4. $8x^2 + 2x = 3$.
5. $x^2 - 7x = 30$.
6. $6x^2 - 11x = 10$.
7. $2x^2 + x = 6$.
8. $x^2 + 14x + 48 = 0$.
9. $6x^2 - x = 5$.
10. $x^2 + 3x - 28 = 0$.
11. $5x^2 - 7x = 0$.

17. $$x^2 - 2x + 1 = 0.$$

Write $$x^2 - 2x + 1 = y.$$

Solving for x, we have

$$x = 1 \pm \sqrt{y}.$$

$y = 0$	$x = 1$
$y = 1$	$x = 2$ and 0
$y = 4$	$x = 3$ and -1
$y = 9$	$x = 4$ and -2
etc.	etc.

Plotting these points and drawing a smooth curve through them, we have the curve shown in Fig. 9. This curve does not cross the axis of X, but touches it at the point (1, 0). The first member of the given equation being a perfect square, the equation has two equal roots. The graph of a quadratic having equal roots always touches the axis of X at a distance from O equal to one of the equal roots.

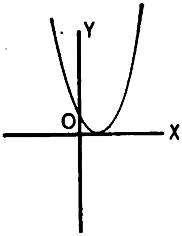

FIG. 9.

Exercises. Draw the graphs of

1. $x^2 - 4x = -4.$
2. $x^2 + 6x = -9.$

GRAPHS OF SIMULTANEOUS QUADRATICS.

18.

1. $x + y = 2.$
2. $xy = -15.$

In the solution as given on page 251, two auxiliary equations occur, viz.

3. $x - y = 8.$
4. $x - y = -8.$

In Fig. 10 the various lines of the graph are numbered to correspond with the numbers of the equations.

Equations (1) and (2) give a straight line and the double-branched curve known as a hyperbola. These intersect at the points P, Q, whose coördinates are $(-3, 5)$ and $(5, -3)$. These are the only solutions to the system of equations. The auxiliary lines (3) and (4) intersect line (1) in P and Q, and hyperbola (2) in R and S. R and S are solutions of the system (3), (4), and (2), but not of the system (1), (2).

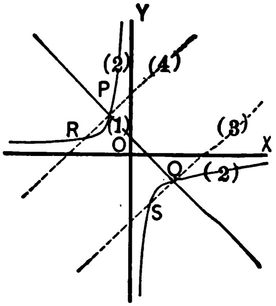

FIG. 10.

19.

1. $xy = 12$.
2. $x^2 + y^2 = 40$.

The auxiliary equations appearing in the solution are:

3. $x + y = +8$.
4. $x + y = -8$.
5. $x - y = +4$.
6. $x - y = -4$.

The graphs of all these equations are shown in Fig. 11 by the corresponding numbers.

The solutions are at the points *P*, *Q*, *R*, and *S*.

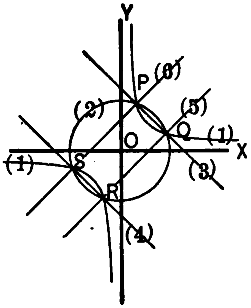

Fig. 11.

Exercises. Draw the graphs of the equations:

Page 251, Examples 3, 4.
Page 253, Examples, 5, 12, 17.
Page 254, Examples, 3, 6.

10

197 11-15

Problems of work

State - 1 no. hrs. alone
2 part work per hr
3 no hours work

A	B
18	24
$\frac{1}{18}$	$\frac{1}{24}$
d	d

Equation: sum am't actually worked = whole am't work (or 1)

$$\frac{d}{18} + \frac{d}{24} = 1$$

Interest

$$\begin{cases} P \\ R \\ T \\ I \end{cases}$$

$I = P \cdot R \cdot T$

$A = P + I$.